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I Theory of particles with arbitrary spin II Photoproduction of Tm173

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I. THEORY OF PARTICLES WITH ARBITRARY SPIN

II. PHOTOPRODUCTION OF Tm^{173}

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PART ONE

THEORY OF PARTICLES WITH ARBITRARY SPIN

I. INTRODUCTION

The history of relativistic wave equations began in 1926 when the Klein-Gordon equation, the first relativistic wave equation, was proposed independently by several authors. This equation is derived by making the operator substitutions $E \to i\hbar \frac{\partial}{\partial t}$, $\vec{p} \to -i\hbar \vec{\nabla}$ in the relativistic relation between the energy and momentum for a free particle,

$$E^2 = c^2 p^2 + m^2 c^4 , \tag{1}$$

so that

$$-\hbar^2 \frac{\partial^2 \phi(x)}{\partial t^2} = \left(-\hbar^2 c^2 \nabla^2 + m^2 c^4\right) \phi(x) . \tag{2}$$

Here $x \sim (\vec{x}, t)$, m is the mass of the particle and c the velocity of light in vacuum. The amplitude $\phi(x)$ is a one-component scalar quantity which, under an inhomogeneous proper Lorentz transformation

$$x'_\mu = a_{\mu\nu} x_\nu + b_\mu , \tag{3}$$

transforms according to

$$\phi'(x') = \phi(x) . \tag{4}$$

To give a physical interpretation to $\phi(x)$ one might try to define a density and a current so that a continuity equation holds between them

in analogy with non-relativistic theory. The density to which one is led is

$$\rho = \frac{i\hbar}{2mc^2}\left(\phi^c \frac{\partial \phi}{\partial t} - \frac{\partial \phi^c}{\partial t}\phi\right), \tag{5}$$

where the superscript c indicates complex conjugate. It is clear that ρ assumes negative as well as positive values because $\frac{\partial \phi}{\partial t}$ can have the opposite sign from ϕ. Thus it is difficult to think of ρ as a conventional probability density.

In 1928 while trying to overcome these difficulties with the interpretation of the Klein-Gordon theory, Dirac (1) discovered the relativistic equation which now bears his name. The reasoning involved in the discovery of the Dirac equation is as follows:

1. To prevent the occurrence of negative probability densities the differential equation must be first order in time derivatives. This gives densities of the form $\phi^c\phi$ which are positive definite.
2. Relativistic covariance implies that there be symmetry in the treatment of the space and time derivatives. Therefore, only first-order spatial derivatives are considered.
3. Linearity is required so that the superposition principle of quantum mechanics holds.
4. The wave function must satisfy the Klein-Gordon equation since it implies the relativistic energy momentum relationship for a free particle.

These considerations lead to the equation

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar} \right) \psi(x) = 0 \,, \tag{6}$$

where repeated indices are summed and Greek indices run from one to four. The Dirac matrices, γ_μ , are 4 x 4 numerical matrices which satisfy the anti-commutation rules

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 \delta_{\mu\nu} \,. \tag{7}$$

Any set of four matrices which satisfy Equation 7 is a possible set of Dirac matrices. The Dirac equation describes a free particle with mass m and spin one-half. The wave function $\psi(x)$ is a four-component quantity which transforms like an elementary spinor with respect to inhomogeneous proper Lorentz transformations. Equation 6 can also be written in the form of two spinor differential equations connecting the elementary spinors $\chi_{\dot{b}}$ and φ^a (the indices ranging from 1 to 2)

$$\partial_{a\dot{b}} \varphi^a = i \frac{mc}{\hbar} \chi_{\dot{b}} \,, \tag{8a}$$

$$\partial^{a\dot{b}} \chi_{\dot{b}} = i \frac{mc}{\hbar} \varphi^a \,. \tag{8b}$$

The spinor differential operator $\partial_{a\dot{b}}$ is related to ordinary differentiation by

$$\partial_{11} = \frac{\partial}{\partial x_3} - i\frac{\partial}{\partial x_4}, \tag{9a}$$

$$\partial_{12} = \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}, \tag{9b}$$

$$\partial_{21} = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}, \tag{9c}$$

$$\partial_{22} = -\frac{\partial}{\partial x_3} - i\frac{\partial}{\partial x_4}, \tag{9d}$$

where $x_4 = ict$.

The success of the Dirac equation for spin one-half makes it plausible to generalize the spinor differential equations to equations for arbitrary spin. This was done by Dirac (2) and Fierz and Pauli (3, 4). They wrote Lorentz covariant first-order differential equations relating two independent spinors each with $k+l+1$ indices:

$$\partial_{a\dot{b}}\, \varphi^{a a_1 \cdots a_l}_{\dot{b}_1 \cdots \dot{b}_k} = i\frac{mc}{\hbar}\, \chi^{a_1 \cdots a_l}_{\dot{b}\dot{b}_1 \cdots \dot{b}_k}, \tag{10a}$$

$$\partial^{a\dot{b}}\, \chi^{a_1 \cdots a_l}_{\dot{b}\dot{b}_1 \cdots \dot{b}_k} = i\frac{mc}{\hbar}\, \varphi^{a a_1 \cdots a_l}_{\dot{b}_1 \cdots \dot{b}_k} . \tag{10b}$$

All the spinor components satisfy the Klein-Gordon equation and the equations describe particles with maximum spin $s = \frac{1}{2}(k+l+1)$. The symmetry of the spinors limits the equations to a single spin.

Equation 6 is a useful form for the spin one-half equations and suggests that the spinor equations for higher spins be rewritten in the form of a first-order matrix differential equation. Kemmer (5) did this for spin zero and spin one, the equation being

$$\left(\beta_\mu \frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar}\right)\psi(x) = 0 . \qquad (11)$$

The β_μ are the Duffin-Kemmer matrices (6). They satisfy the algebraic relations:

$$\beta_\mu \beta_\nu \beta_\lambda + \beta_\lambda \beta_\nu \beta_\mu = \beta_\mu \delta_{\nu\lambda} + \beta_\lambda \delta_{\nu\mu} . \qquad (12)$$

The β_μ have two irreducible representations, five-dimensional for spin zero and ten-dimensional for spin one. This means that the wave function solutions have five and ten components respectively for the two cases. The Kemmer equations are special cases of the general type proposed by Bhabha (7) for arbitrary spin. The Bhabha equations have the same form as Equation 11 but with the β_μ chosen properly for each spin. These equations are equivalent to the Dirac-Fierz-Pauli equations.

Another general scheme for describing relativistic free particles is the helicity representation of Jacob and Wick (8). This representation was introduced to facilitate the analysis of binary reactions involving particles with arbitrary spin and does not involve the explicit writing of a wave equation or the explicit form of the state functions.

Foldy and Wouthuysen (9) by the unitary transformation which bears

their name were able to rewrite the Dirac equation in a form suitable for taking the non-relativistic limit. Using the Hamiltonian form proposed by Taketani and Sakata (10), Case (11) showed that the Klein-Gordon and spin one Dirac-Fierz-Pauli theory could be written in the Foldy-Wouthuysen form. This led Foldy (12) to propose a relativistic wave equation for arbitrary spin of the form

$$\omega \beta \varphi(x) = i\hbar \frac{\partial \varphi(x)}{\partial t} \quad , \tag{13}$$

where $\omega = \sqrt{m^2 - \hbar^2 \nabla^2}$ and β is a $2\times(2A+1)$ rank square matrix which is the generalization to arbitrary spin of the γ_4 matrix of spin one-half theory. A similar equation has been recently proposed by Guth (13) but in a more obviously covairant form.

Recently the Foldy-Wouthuysen transformation for spin one-half has been interpreted (14, 15) as the Lorentz transformation to the rest system of the particle. The Foldy-Wouthuysen transforms of the laboratory system Hamiltonian and polarization operators are thus to be interpreted as the operators Lorentz transformed to the rest system. In the rest system these spin one-half operators are β and $\beta \frac{\vec{\sigma}}{2} \cdot \hat{e}$ where β is γ_4, $\hat{e}$ is an arbitrary unit vector, and $\vec{\sigma}$ are the Pauli matrices. This interpretation of the Foldy-Wouthuysen transformation, generalized to arbitrary spin, is used in this thesis to construct a theory of particles with mass and arbitrary spin.

The theory proposed in this thesis is therefore derived from two postulates:

1. Rest system energy and polarization operators exist for all spins with a form analogous to those for spin one-half. That is, for arbitrary spin the rest system energy and polarization operators are β and $\beta \vec{S}\cdot\hat{\vec{e}}$ where β is the generalization of γ_4 and $\vec{S}$ is a $2s(2s+1)$ rank matrix with the spin matrices $\vec{J}$ along the diagonal and zeros elsewhere.
2. Laboratory system spinors and laboratory system operators can be constructed from the rest system operators by a generalization of the spin one-half Lorentz transformation out of the rest system.

As a result of these postulates laboratory Hamiltonians and polarization operators can be calculated for all spins as well as the $2s(2s+1)$ component plane wave eigenstates. The other displacement operators such as the momentum and angular momentum are also specified for arbitrary spin. Thus a complete Hamiltonian type relativistic theory for particles of arbitrary spin is obtained.

An interesting feature of the theory is that it goes continuously over to the massless particle theory of Hammer and Good (16) which contains the two-component neutrino theory and the photon as well as all higher spins. Another feature is that the spin zero and spin one laboratory Hamiltonians are different from those proposed by Taketani and Sakata (10). It is therefore possible that interactions will appear in this theory in a manner different from previous proposals.

II. FORMULATION OF THE THEORY

A. Construction of Spinors

Any description of a particle and antiparticle with integer or half-integer spin $\mathcal{A}$ and mass requires $2\times(2\mathcal{A}+1)$ independent functions, $2\mathcal{A}+1$ spin projections for both particle and antiparticle. Postulate one of the arbitrary spin theory is that β and $\beta\vec{S}\cdot\hat{e}$ are the rest system energy and polarization operators. Thus, the $2\times(2\mathcal{A}+1)$ independent rest system functions are the eigenfunctions $\psi^{0}_{\epsilon,m_{\mathcal{A}}}(\hat{e})$ of the matrix equations

$$\beta\psi^{0}_{\epsilon,m_{\mathcal{A}}}(\hat{e}) = \epsilon\psi^{0}_{\epsilon,m_{\mathcal{A}}}(\hat{e}), \tag{14a}$$

$$\beta\vec{S}\cdot\hat{e}\,\psi^{0}_{\epsilon,m_{\mathcal{A}}}(\hat{e}) = m_{\mathcal{A}}\psi^{0}_{\epsilon,m_{\mathcal{A}}}(\hat{e}), \tag{14b}$$

where

$$\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{15a}$$

$$\vec{S} = \begin{pmatrix} \vec{\mathcal{A}} & 0 \\ 0 & \vec{\mathcal{A}} \end{pmatrix}, \tag{15b}$$

each submatrix being a $2\mathcal{A}+1$ rank square matrix. The operators $\vec{\mathcal{A}}$ are Hermitian spin matrices for spin $\mathcal{A}$ and $\epsilon=\pm1$ for particles/antiparticles. The eigenfunctions can be written as

$$\psi^{o}_{\epsilon,m_s}(\hat{e}) = \frac{1}{\sqrt{2}} \begin{pmatrix} u(\hat{e})_{\epsilon m_s} \\ \epsilon\, u(\hat{e})_{\epsilon m_s} \end{pmatrix}, \tag{16a}$$

where

$$\vec{S}\cdot\hat{e}\; u(\hat{e})_{\epsilon m_s} = \epsilon m_s u(\hat{e})_{\epsilon m_s} \tag{16b}$$

and $m_s = -s, -s+1, \cdots, s-1, s$. The superscript o on a quantity refers to the rest system value of the quantity. The completeness of the $\psi^{o}(\hat{e})_{\epsilon,m_s}$ is expressed by

$$\sum_{\epsilon,m_s} \left(\psi^{o}(\hat{e})_{\epsilon,m_s}\right)_a \left(\psi^{o}(\hat{e})_{\epsilon,m_s}\right)^{c}_{b} = \delta_{ab} . \tag{16c}$$

The rest system wave equation from which Equation 14a follows is

$$m\beta \Psi^{o}(x^o)_{\epsilon,m_s} = i\frac{\partial}{\partial t} \Psi^{o}(x^o)_{\epsilon,m_s} , \tag{17}$$

where

$$\Psi^{o}(x^o)_{\epsilon,m_s} = \psi^{o}(\hat{e})_{\epsilon,m_s} e^{-i\epsilon m t^o} . \tag{18}$$

The units used here and subsequently are $\hbar = c = 1$.

The laboratory system eigenfunctions are obtained from $\Psi^{0}_{\epsilon i m_s}(x^0)$ by using postulate two. To see this consider the laboratory system in a state of definite energy $E > 0$ and momentum $\vec{q}$. The special Lorentz transformation between the rest and laboratory coordinate systems is then (17)

$$x_\mu = a_{\mu r} x_r^0 \quad , \tag{19}$$

where $x_\mu \sim (\vec{x}, it)$, and

$$a_{ij} = \delta_{ij} + \frac{q_i q_j}{m(E+m)} \quad , \tag{20a}$$

$$a_{i4} = -a_{4i} = i\frac{q_i}{m} \quad , \tag{20b}$$

$$a_{44} = E/m \quad . \tag{20c}$$

The corresponding wave function transformation is by postulate two

$$\Psi_{\epsilon i m_s}(x) = e^{\hat{q}\cdot\vec{\alpha}\,\operatorname{arctanh} q/E}\,\Psi^{0}_{\epsilon i m_s}(x^0) \tag{21}$$

for all spins in analogy with the spin one-half case, where

$$\vec{\alpha} \equiv \begin{pmatrix} \vec{\Delta} & 0 \\ 0 & -\vec{\Delta} \end{pmatrix} = -\gamma_5 \vec{S} , \tag{22a}$$

$$\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{22b}$$

The exponential factor $e^{-i\epsilon m t^0}$ is a Lorentz scalar whose laboratory system value is $e^{i\epsilon q_\mu x_\mu}$ where $q_4 = iE$. It is convenient to rewrite this exponential factor and the transformation operator of Equation 21 in terms of $\vec{P}$ the eigenvalue of $-i\vec{\nabla}$:

$$e^{\hat{\vec{q}}\cdot\vec{\alpha}\,\mathrm{arctanh}\,q/E} \quad e^{\epsilon\hat{\vec{P}}\cdot\vec{\alpha}\,\mathrm{arctanh}\,P/E}$$
$$= \equiv \Lambda(\epsilon\vec{P}) , \tag{23a}$$

$$e^{i\epsilon q_\mu x_\mu} = e^{i(\vec{P}\cdot\vec{x} - \epsilon E t)} . \tag{23b}$$

Note that $\vec{q}$ is the eigenvalue of $\frac{P_0}{|P_0|}(-i\vec{\nabla})$ where $P_0 \equiv i\frac{\partial}{\partial t}$, so $\vec{q} = \epsilon\vec{P}$. This is the generalization to arbitrary spin of the electron-positron interpretation of the solution of the Dirac equation. That is, for the free Dirac particle, as an alternative to Dirac's hole theory, one may interpret the four solutions of the Dirac equation as actually describing electrons and positrons instead of electrons alone. The observable, and therefore physical, momentum and energy are then

$|H|$ and $\frac{H}{|H|}\vec{P}$. Thus the positron has the energy $-H$ and momentum $-\vec{P}$. In the generalization to arbitrary spin the particle has the same physical assignments as the electron and the antiparticle the same physical assignments as the positron.

The most general state of the laboratory system is given by a superposition of the plane wave eigenstates,

$$\Psi(x) = \frac{m^{\lambda}}{(2\pi)^{3/2}} \int \frac{d\vec{P}}{E} \sum_{\epsilon, m_s} A(\vec{P})_{\epsilon, m_s} \Lambda(\epsilon \vec{P}) \psi^{\circ}(\hat{e})_{\epsilon, m_s} e^{i(\vec{P}\cdot\vec{x} - \epsilon E t)} \quad . \tag{24}$$

The factor E^{-1} is included because $\frac{d\vec{P}}{E}$ is a Lorentz scalar. The inclusion of the factor m^{λ} will be shown later to make the theory go continuously to the correct massless limit. For larger powers of m the functions Ψ go to zero as m goes to zero, and for smaller powers of m the function Ψ goes to infinity as m goes to zero. Notice that $\Psi(x)$ can be written as

$$\Psi(x) = m^{\lambda} \omega^{-\frac{1}{2}} \Lambda(x) \varphi(x) \quad , \tag{25}$$

where $\Lambda(x)$ is the operator generalization of $\Lambda(\epsilon \vec{P})$ formed by replacing $\vec{P}$ with $-i\vec{\nabla}$ and ϵ with $P_0/|P_0|$, and

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{P}}{\sqrt{E}} \sum_{\epsilon, m_s} A(\vec{P})_{\epsilon, m_s} \psi^{\circ}(\hat{e})_{\epsilon, m_s} e^{i(\vec{P}\cdot\vec{x} - \epsilon E t)} \quad . \tag{26}$$

Since $\varphi(x)$ contains only the column matrices $\psi^{\circ}(\hat{e})_{\sigma_1 m_2}$, it follows from Equation 14a that $\varphi(x)$ satisfies Equation 13. Also, as will be shown later (See Chapter IV), $\varphi(x)$ has the Lorentz transformation properties discussed by Foldy (12) and so is the Foldy wave function.

B. Covariance of the Theory

The theory defined by the two postulates of the previous section will be shown to be covariant. This is a necessary requirement since all physical theories should have the same form in coordinate systems related by proper Lorentz transformations. This means that if $\Psi(x)$ satisfies Equation 24, then the Lorentz transformed laboratory system function $\Psi(x')$ is

$$\Psi'(x') = \frac{m^2}{(2\pi)^{3/2}} \int \frac{d\vec{p}'}{E'} \sum_{\sigma_1 m_2} A'(\vec{p}')_{\sigma_1 m_2} \Lambda(\epsilon \vec{p}') \psi^{\circ}(\hat{e})_{\sigma_1 m_2} e^{i(\vec{p}'\cdot\vec{x}' - \epsilon E' t')} \tag{27}$$

for the theory to be covariant. The function $\Psi(x)$ is a spinor with respect to proper Lorentz transformations between laboratory systems. This will be verified by demonstrating the covariance using spinor transformation rules for $\Psi(x)$. The special Lorentz transformation between laboratory systems with relative velocity $\vec{v}$ is (17)

$$x'_\mu = a_{\mu\nu} x_\nu \ , \tag{28a}$$

where

$$a_{ij} = \delta_{ij} + (\gamma - 1)\frac{v_i v_j}{v^2} \quad , \tag{28b}$$

$$a_{i4} = -a_{4i} = i\gamma v_i \quad , \tag{28c}$$

$$a_{44} = \gamma \quad , \tag{28d}$$

and $\gamma = (1 - v^2)^{-\frac{1}{2}}$. The corresponding spinor transformation is

$$\Psi'(x') = e^{i\vec{\tau}\cdot\vec{\alpha}}\,\Psi(x) \quad , \tag{29}$$

where $\vec{\tau}$ is an imaginary vector defined by

$$\vec{\tau} = i\,\hat{\vec{v}}\ \text{arctanh}\ v \quad . \tag{30}$$

Equation 21 is a special case of Equation 29 with $\vec{\tau} = -i\hat{\vec{\beta}}\,\text{arctanh}\,p/E$.

For the pure rotation

$$x_\mu' = b_{\mu\nu}x_\nu \quad , \tag{31a}$$

where

$$b_{ij} = \delta_{ij}\cos\tau + \epsilon_{ijk}\hat{\tau}_k \sin\tau + \hat{\tau}_i\hat{\tau}_j(1-\cos\tau), \tag{31b}$$

$$b_{i4} = b_{4i} = 0 , \tag{31c}$$

$$b_{44} = 1 , \tag{31d}$$

the spinor transformation rule is

$$\Psi'(x') = e^{i\vec{\tau}\cdot\vec{S}}\Psi(x) , \tag{32}$$

where $\vec{\tau}$ is real. The transformation is an angular displacement in the right-handed sense about $\hat{\vec{\tau}}$ through angle τ $(\tau^2 \equiv \tau_i\tau_i)$. Equations 29 and 32 can be written as

$$\Psi'(x') = \begin{pmatrix} e^{i\vec{\tau}\cdot\vec{\lambda}} & 0 \\ 0 & e^{i\vec{\tau}^c\cdot\vec{\lambda}} \end{pmatrix}\Psi(x) . \tag{33}$$

The product of two successive transformations is again a transformation with the form of Equation 33. That is,

$$\begin{pmatrix} e^{i\vec{\tau}_A\cdot\vec{\lambda}} & 0 \\ 0 & e^{i\vec{\tau}^c_A\cdot\vec{\lambda}} \end{pmatrix}\begin{pmatrix} e^{i\vec{\tau}_B\cdot\vec{\lambda}} & 0 \\ 0 & e^{i\vec{\tau}^c_B\cdot\vec{\lambda}} \end{pmatrix} = \begin{pmatrix} e^{i\vec{\tau}_{AB}\cdot\vec{\lambda}} & 0 \\ 0 & e^{i\vec{\tau}^c_{AB}\cdot\vec{\lambda}} \end{pmatrix}. \qquad (34)$$

Equation 34 is proved by a theorm of Hansdorff (18) which states that

$$e^{A}\,e^{B} = e^{A+B+\frac{1}{2}[A,B]_- + \frac{1}{12}[A,[A,B]_-]_- + \cdots}, \qquad (35)$$

where the higher order terms in the series depend only on successive higher order commutators. The commutation rules for all spins are

$$[\lambda_i, \lambda_j]_- = i\epsilon_{ijk}\lambda_k . \qquad (36)$$

Thus each term in Equation 35 with $A = i\vec{\tau}_A\cdot\vec{\lambda}$ $(A = i\vec{\tau}^c_A\cdot\vec{\lambda})$ and $B = i\vec{\tau}_B\cdot\vec{\lambda}$ $(B = i\vec{\tau}^c_B\cdot\vec{\lambda})$ is linear in λ_k and the terms can be summed to give $i\vec{\tau}_{AB}\cdot\vec{\lambda}$ $(i\vec{\tau}^c_{AB}\cdot\vec{\lambda})$.

As a consequence of Equation 34, every Lorentz transformation continuous with the identity can be generated by the two types discussed in the previous paragraph. It is therefore sufficient to prove covariance separately for the pure rotations and special Lorentz transformations.

For pure rotations, Equation 32 becomes

$$\Psi'(x') = \frac{m_A}{(2\pi)^{3/2}} \int \frac{d\vec{p}}{E} \sum_{\epsilon, m_A} A(\vec{p})_{\epsilon, m_A} e^{i\vec{\tau}\cdot\vec{S}} \Lambda(\epsilon\vec{p}) e^{-i\vec{\tau}\cdot\vec{S}} e^{i\vec{\tau}\cdot\vec{S}} \Psi^{\circ}_{\epsilon, m_A}(\hat{\vec{e}}) e^{i(\vec{p}\vec{x} - \epsilon E t)} \quad (37)$$

The evaluation of $e^{i\vec{\tau}\cdot\vec{S}} \Lambda(\epsilon\vec{p}) e^{-i\vec{\tau}\cdot\vec{S}}$ and $e^{i\vec{\tau}\cdot\vec{S}} \Psi^{\circ}_{\epsilon, m_A}(\hat{\vec{e}})$ follows from the identity

$$e^{-i\vec{\tau}\cdot\vec{S}} S_i e^{i\vec{\tau}\cdot\vec{S}} = a_{ik} S_k , \quad (38)$$

which is proved in Appendix I. It is easy to see using Equation 38 that

$$e^{i\vec{\tau}\cdot\vec{S}} \vec{S}\cdot\hat{\vec{e}} e^{-i\vec{\tau}\cdot\vec{S}} = \vec{S}\cdot\hat{\vec{e}}' , \quad (39)$$

where $\hat{e}'_i = b_{ij} \hat{e}_j$. Since the matrix β commutes with $\vec{S}$ the result of a rotation on $\Psi^{\circ}_{\epsilon, m_A}(\hat{\vec{e}})$ is

$$e^{i\vec{\tau}\cdot\vec{S}} \Psi^{\circ}_{\epsilon, m_A}(\hat{\vec{e}}) = e^{i\delta_{\epsilon, m_A}} \Psi^{\circ}_{\epsilon, m_A}(\hat{\vec{e}}') , \quad (40)$$

where $\delta_{\epsilon,m_\Delta}$ is real. The rotated functions $e^{i\vec{\tau}\cdot\vec{S}}\psi^0_{\epsilon,m_\Delta}(\hat{\vec{e}})$ can also be written in terms of the complete set of functions $\psi^0_{\epsilon,m_\Delta}(\hat{\vec{e}})$ according to

$$e^{i\vec{\tau}\cdot\vec{S}}\psi^0_{\epsilon,m_\Delta}(\hat{\vec{e}}) = \sum_{m'_\Delta}\psi^0_{\epsilon,m'_\Delta}(\hat{\vec{e}})\,D^{\Delta}_{m'_\Delta,m_\Delta}(\vec{\tau}), \tag{41}$$

where

$$D^{\Delta}_{m'_\Delta,m_\Delta}(\vec{\tau}) = \psi^{0\,H}_{\epsilon,m'_\Delta}(\hat{\vec{e}})\,e^{i\vec{\tau}\cdot\vec{S}}\,\psi^0_{\epsilon,m_\Delta}(\hat{\vec{e}}) \tag{42}$$

are the usual matrix elements (19) for finite rotations. The superscript H indicates Hermitian conjugate. Also from Equation 38

$$e^{i\vec{\tau}\cdot\vec{S}}\,\Lambda(\epsilon\vec{p})\,e^{-i\vec{\tau}\cdot\vec{S}} = \Lambda(\epsilon\vec{p}'), \tag{43}$$

because $\Lambda(\epsilon\vec{p})$ is a polynomial in $\hat{\vec{p}}\cdot\vec{\alpha}$ (see Appendix II). Substituting Equations 41 and 43 into Equation 37 gives

$$\Psi'(x') = \frac{m^{\Delta}}{(2\pi)^{3/2}}\int\frac{d\vec{p}}{E}\sum_{\epsilon,m_\Delta}\sum_{m'_\Delta}A(\vec{p})_{\epsilon,m_\Delta}\,\Lambda(\epsilon\vec{p}')\,D^{\Delta}_{m'_\Delta,m_\Delta}\,\psi^0_{\epsilon,m'_\Delta}(\hat{\vec{e}})\,e^{i(\vec{p}\cdot\vec{x}-\epsilon Et)}. \tag{44}$$

A change of integration variable can now be made so that $d\vec{P}' = d\vec{P}$, $E' = E$ and $\vec{P}' \cdot \vec{x}' = \vec{P} \cdot \vec{x}$. Comparison to Equation 27 demonstrates the covariance of $\Psi(x)$ and give the transformation rule for the expansion coefficients

$$A'(\vec{P}')_{\epsilon i m_\Lambda'} = \sum_{m_\Lambda} D^{\Lambda}_{m_\Lambda' m_\Lambda} A(\vec{P})_{\epsilon i m_\Lambda} . \tag{45}$$

For pure special Lorentz transformations, Equation 29 becomes

$$\Psi'(x') = \frac{m^\Lambda}{(2\pi)^{3/2}} \int \frac{d\vec{P}}{E} \sum_{\epsilon i m_\Lambda} A(\vec{P})_{\epsilon i m_\Lambda} e^{i\vec{\tau}\cdot\vec{\alpha}} \Lambda(\epsilon\vec{P}) \Psi^0_{\epsilon i m_\Lambda}(\hat{\vec{e}}) e^{i(\vec{P}\cdot\vec{x} - \epsilon E t)} . \tag{46}$$

As proved in Appendix I,

$$e^{i\vec{\tau}\cdot\vec{\alpha}} \Lambda(\epsilon\vec{P}) = \Lambda(\epsilon\vec{P}') e^{i\vec{\tau}_R \cdot \vec{S}} , \tag{47}$$

where $\vec{\tau}$ is given by Equation 30. The parameter $\vec{\tau}_R$ specifies a particular pure rotation with the axis of rotation in the direction of $\epsilon \vec{P} \times \vec{V}$ and with magnitude

$$\tau_R = 2 \arctan\left[\frac{\gamma |\frac{\vec{P}}{m} \times \vec{V}|}{(\gamma+1)(\frac{E}{m}+1) - \gamma \epsilon \frac{\vec{P}}{m} \cdot \vec{V}} \right] . \tag{48}$$

A change of integration variable parallel to the coordinate transformation can be made using the identities $\frac{d\vec{p}}{E} = \frac{d\vec{p}'}{E'}$ and $p_\mu x_\mu = p'_\mu x'_\mu$. With Equation 47 substituted, Equation 46 for $\Psi'(x')$ then becomes

$$\Psi'(x') = \frac{m_2}{(2\pi)^{3/2}} \int \frac{d\vec{p}'}{E'} \sum_{\epsilon_1 m_2} A(\vec{p}) \Lambda(\epsilon \vec{p}') e^{i\vec{\tau}_R \cdot \vec{S}} \Psi^0_{\epsilon_1 m_2}(\hat{e}) e^{i(\vec{p}'\cdot\vec{x}' - \epsilon E' t')} \quad (49)$$

Again the expansion in Equation 41 is substituted and the covariance is demonstrated with

$$A'_{\epsilon_1 m'_2}(\vec{p}') = \sum_{m_2} D^{\frac{1}{2}}_{m'_2 m_2}(\vec{\tau}_R) A_{\epsilon_1 m_2}(\vec{p}) \quad . \quad (50)$$

The reflection properties of the theory must also be considered. With respect to the space reflection

$$x_i' = -x_i \quad , \quad x_4' = x_4 \quad , \quad (51a)$$

and the time reflection

$$x_i' = x_i \quad , \quad x_4' = -x_4 \quad , \quad (51b)$$

the transformation of the spinor components is

$$\Psi'(x') = \Lambda_S \left[C \bar{\Psi}(x) \right]^C , \tag{52a}$$

$$\Psi'(x') = \Lambda_T \left[C \bar{\Psi}(x) \right]^C , \tag{52b}$$

with

$$\Lambda_S = i \beta , \tag{53a}$$

$$\Lambda_T = \gamma_5 \beta , \tag{53b}$$

in analogy with the spin one-half theory. The matrix γ_5 is defined by

$$\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} , \tag{54}$$

each submatrix being $2l+1$ dimensional. In the representation of the spin matrices

$$(J_1)_{m_l+1, m_l} = (J_1)_{m_l, m_l+1} = \tfrac{1}{2} \left[(l - m_l)(l + m_l + 1) \right]^{\frac{1}{2}} , \tag{55a}$$

$$(J_2)_{m_l+1, m_l} = -(J_2)_{m_l, m_l+1} = -\tfrac{1}{2} i \left[(l - m_l)(l + m_l + 1) \right]^{\frac{1}{2}} , \tag{55b}$$

$$(\Lambda_3)_{m_\Lambda, m_\Lambda} = m_\Lambda , \tag{55c}$$

the charge conjugation matrix C is defined by

$$C = \begin{pmatrix} 0 & U \\ -U & 0 \end{pmatrix}, \tag{56}$$

where

$$U_{m_\Lambda, m_\Lambda'} = (-1)^{\Lambda - m_\Lambda} \delta_{m_\Lambda, -m_\Lambda'} . \tag{57}$$

This definition is discussed in Appendix III. The matrix U has the properties

$$U \vec{\Lambda} U^{-1} = -\vec{\Lambda}^c , \tag{58a}$$

$$U^{\dagger} = U^{-1} , \tag{58b}$$

and

$$U^{-1} = \pm U , \tag{58c}$$

where $\pm$ applies to bosons/fermions.

Consequently C is real and unitary and

$$C\vec{S}C^{-1} = -\vec{S}^{c}, \tag{59a}$$

$$C\vec{\alpha}C^{-1} = \vec{\alpha}^{c}, \tag{59b}$$

$$C\beta C^{-1} = -\beta. \tag{59c}$$

To verify that the transformation rules give covariance, one starts with the plane wave expansion

$$\Psi'(x') = \frac{m\Delta}{(2\pi)^{3/2}} \int \frac{d\vec{p}}{E} \sum_{\sigma_1 m_\Delta} A^{c}_{\sigma_1 m_\Delta}(\vec{p})\, \Lambda_{S,T}\, C\, \Lambda^{c}(\epsilon\vec{p})\, C^{-1} \Lambda^{-1}_{S,T}$$

$$\times \Lambda_{S,T}\, C\, \Psi^{o\,c}_{\sigma_1 m_\Delta}(\hat{e})\, e^{-i(\vec{p}\cdot\vec{x} - \epsilon E t)}. \tag{60}$$

From the properties of the matrices it is easy to see that

$$\Lambda_{S,T}\, C\, \Lambda^{c}(\epsilon\vec{p})\, C^{-1} \Lambda^{-1}_{S,T} = \Lambda(-\epsilon\vec{p}), \tag{61}$$

and that

$$\Lambda_S C \psi^{0c}_{\epsilon 1 m_\Delta}(\hat{\vec{e}}) = e^{i\eta_\epsilon} \psi^{0}_{-\epsilon 1 m_\Delta}(\hat{\vec{e}}) , \tag{62a}$$

$$\Lambda_T C \psi^{0c}_{\epsilon 1 m_\Delta}(\hat{\vec{e}}) = e^{i\gamma_{m_\Delta}} \psi^{0}_{\epsilon 1 -m_\Delta}(\hat{\vec{e}}) , \tag{62b}$$

where η_ϵ and γ_{m_Δ} are phase factors. For the space reflection substitution of Equations 61 and 62a into 60 gives

$$\Psi'(x') = \frac{m_\Delta}{(2\pi)^{3/2}} \int \frac{d\vec{p}}{E} \sum_{\epsilon 1 m_\Delta} A^{c}_{\epsilon 1 m_\Delta}(\vec{p}) \Lambda(-\epsilon \vec{p}) e^{i\eta_\epsilon} \psi^{0}_{-\epsilon 1 m_\Delta}(\hat{\vec{e}}) e^{-i(\vec{p}\cdot\vec{x} - \epsilon E t)} . \tag{63}$$

Letting $\epsilon \to -\epsilon$ and substituting for $\vec{x}$ from Equation 51a demonstrates the covariance with respect to the space reflection. The transformation rule for the $A_{1\Delta}$ is found to be

$$A'_{\epsilon 1 m_\Delta}(\vec{p}) = e^{i\eta_{-\epsilon}} A^{c}_{-\epsilon 1 m_\Delta}(\vec{p}) . \tag{64}$$

Similarly $\Psi(x)$ covariant with respect to the time reflection with the $A'_{1\Delta}$ transforming according to

$$A'^{(\vec{p})}_{\epsilon_1 m_s} = e^{i\gamma_{m_s}} A^{c}_{\epsilon_1 -m_s}(-\vec{p}) \quad . \tag{65}$$

The theory is also charge conjugate. The spinor transformation rule is

$$\Psi'(x') = [C\Psi(x)]^{c}, \tag{66}$$

and the properties of C lead to

$$C\Psi^{oc}_{\epsilon_1 m_s}(\hat{e}) = e^{i\mu_\epsilon} \Psi^{o}_{-\epsilon_1 m_s}(\hat{e}), \tag{67}$$

where μ_ϵ is a phase factor. The covariance is demonstrated similar to the procedure for the reflections and the resulting transformation of the A_{1s} is

$$A'^{(\vec{p})}_{\epsilon_1 m_s} = e^{i\mu_\epsilon} A^{c}_{-\epsilon_1 m_s}(-\vec{p}) \quad . \tag{68}$$

III. CONSTRUCTION OF LABORATORY SYSTEM OPERATORS

A. Relation Between Lorentz and Foldy-Wouthuysen Transformation

The unitary Foldy-Wouthuysen operator for spin one-half (9) is well-known to be

$$S_{FW} = \frac{E+m-2\vec{p}\cdot\vec{\alpha}\beta}{[2E(E+m)]^{\frac{1}{2}}} . \tag{69}$$

Bollini and Giambiagi (14) and Good and Rose (15) have shown that S_{FW} corresponds to the spin one-half Lorentz transformation from the laboratory to the rest system together with a factor which preserves the norm of the wave function. To show this, consider the spinor Lorentz transformation out of the rest system, Equation 23a. For spin one-half the exponential expands to give (see Appendix II)

$$\Lambda_{\frac{1}{2}}^{(\epsilon\vec{\beta})} = \frac{E+m+2\epsilon\vec{p}\cdot\vec{\alpha}}{[2m(E+m)]^{\frac{1}{2}}} . \tag{70}$$

Consider $\Lambda_{\frac{1}{2}}^{(\epsilon\vec{\beta})}$ operating on a spin one-half rest system function. The index ϵ commutes with $\vec{p}\cdot\vec{\alpha}$ so it can be moved to the right and then replaced by β according to Equation 14a. Thus,

$$\Lambda_{\frac{1}{2}}^{(\epsilon\vec{\beta})}\Psi^{0}_{\epsilon m_s}(x^0) = \frac{[E+m+2\vec{p}\cdot\vec{\alpha}\epsilon]}{[2m(E+m)]^{\frac{1}{2}}}\Psi^{0}_{\epsilon m_s}(x^0)$$

$$= \frac{[E+m+2\vec{p}\cdot\vec{\alpha}\beta]}{[2m(E+m)]^{1/2}} \Psi^{o}_{\epsilon,m_\ell}(x^o)$$

$$= F_{\frac{1}{2}}(\vec{p})\, \Psi^{o}_{\epsilon,m_\ell}(x^o)$$

$$= \left(\frac{E}{m}\right)^{\frac{1}{2}} S_{FW}^{-1}\, \Psi^{o}_{\epsilon,m_\ell}(x^o), \tag{71}$$

where

$$F_{\frac{1}{2}}(\vec{p}) = \frac{E+m+2\vec{p}\cdot\vec{\alpha}\beta}{[2m(E+m)]^{1/2}}. \tag{72}$$

Next consider a laboratory system function, Equation 21, and $\Lambda^{-1}_{\frac{1}{2}}(\epsilon\vec{p})$ which is

$$\Lambda^{-1}_{\frac{1}{2}}(\epsilon\vec{p}) = \frac{E+m-2\epsilon\vec{p}\cdot\vec{\alpha}}{[2m(E+m)]^{1/2}}. \tag{73}$$

Performing a similar substitution one gets

$$\Lambda^{-1}_{\frac{1}{2}}(\epsilon\vec{p})\,\Psi^{(x)}_{\epsilon,m_s} = \frac{[E+m-2\vec{p}\cdot\vec{\alpha}\epsilon]}{[2m(E+m)]^{\frac{1}{2}}}\,\Psi^{(x)}_{\epsilon,m_s}$$

$$= \frac{[E+m-2\vec{p}\cdot\vec{\alpha}\,H/|H|]}{[2m(E+m)]^{\frac{1}{2}}}\,\Psi^{(x)}_{\epsilon,m_s}$$

$$= F^{-1}_{\frac{1}{2}}(\vec{p})\,\Psi^{(x)}_{\epsilon,m_s}, \tag{74}$$

where $H/|H|$ is the Dirac laboratory system sign operator

$$H/|H| = \frac{2\vec{p}\cdot\vec{\alpha}+m\beta}{E}\,. \tag{75}$$

Hence,

$$\Lambda^{-1}_{\frac{1}{2}}(\epsilon\vec{p})\,\Psi^{(x)}_{\epsilon,m_s} = \frac{m}{E}\,\frac{[E+m-2\vec{p}\cdot\vec{\alpha}\beta]}{[2m(E+m)]^{\frac{1}{2}}}\,\Psi^{(x)}_{\epsilon,m_s}$$

$$= \left(\frac{m}{E}\right)^{\frac{1}{2}} S_{FW}\,\Psi^{(x)}_{\epsilon,m_s}\,. \tag{76}$$

Thus, to relate the Lorentz transformation out of the rest system to the

Foldy-Wouthuysen transformation, $F_{\frac{1}{2}}(\vec{p})$, replace ϵ , which commutes with all operators, by the rest system sign operator β which does not commute with all operators. Similarly, to relate the Lorentz transformation to the rest system to the inverse Foldy-Wouthuysen operator, $F^{-1}_{\frac{1}{2}}(\vec{p})$, replace ϵ by $H/|H|$.

The spin-one spinor Lorentz transformation out of the rest system may be related to a spin one Foldy-Wouthuysen operator $F_1(\vec{p})$ in a similar way. The spin one Lorentz transformation out of the rest system is (see Appendix II)

$$\Lambda_1(\epsilon\vec{p}) = \left(\frac{E}{m}-1\right)(\hat{\vec{p}}\cdot\vec{\alpha})^2 + \epsilon\frac{\vec{p}}{m}\cdot\vec{\alpha} + 1 \ . \tag{77}$$

One can replace ϵ in $\Lambda_1(\epsilon\vec{p})$ by β when it is operating on a spin one rest system eigenfunction so that

$$\Lambda_1(\epsilon\vec{p})\,\Psi^0_{\epsilon 1 m_\ell}(x^0) = \left[\left(\frac{E}{m}-1\right)(\hat{\vec{p}}\cdot\vec{\alpha})^2 + \frac{\vec{p}}{m}\cdot\vec{\alpha}\,\epsilon + 1\right]\Psi^0_{\epsilon 1 m_\ell}(x^0)$$

$$= \left[\left(\frac{E}{m}-1\right)(\hat{\vec{p}}\cdot\vec{\alpha})^2 + \frac{\vec{p}}{m}\cdot\vec{\alpha}\,\beta + 1\right]\Psi^0_{\epsilon 1 m_\ell}(x^0)$$

$$= F_1(\vec{p})\,\Psi^0_{\epsilon 1 m_\ell}(x^0) \ , \tag{78}$$

where

$$F_1(\vec{p}) \equiv \left(\frac{E}{m}-1\right)(\hat{p}\cdot\vec{\alpha})^2 + \frac{\vec{p}}{m}\cdot\vec{\alpha}\,\beta + 1 \tag{79}$$

is defined as the spin one Foldy-Wouthuysen operator. Next consider a spin one laboratory system function, Equation 21, and $\Lambda_1^{-1}(\epsilon\vec{p})$ which is

$$\Lambda_1^{-1}(\epsilon\vec{p}) = \left(\frac{E}{m}-1\right)(\hat{p}\cdot\vec{\alpha})^2 - \epsilon\frac{\vec{p}}{m}\cdot\vec{\alpha} + 1\,. \tag{80}$$

Following the spin one-half procedure,

$$\Lambda_1^{-1}(\epsilon\vec{p})\underset{\epsilon 1 m_a}{\Psi(x)} = \left[\left(\frac{E}{m}-1\right)(\hat{p}\cdot\vec{\alpha})^2 - \frac{\vec{p}}{m}\cdot\vec{\alpha}\,\epsilon + 1\right]\underset{\epsilon 1 m_a}{\Psi(x)}$$

$$= \left[\left(\frac{E}{m}-1\right)(\hat{p}\cdot\vec{\alpha})^2 - \frac{\vec{p}}{m}\cdot\vec{\alpha}\,\frac{H}{|H|} + 1\right]\underset{\epsilon 1 m_a}{\Psi(x)}$$

$$= F_1^{-1}(\vec{p})\underset{\epsilon 1 m_a}{\Psi(x)}, \tag{81}$$

where $H/|H|$ is to be interpreted as the laboratory system sign operator for spin one. That $F_1^{-1}(\vec{p})$ is indeed the matrix inverse of $F_1(\vec{p})$ is

true because F_1 and F_1^{-1} were constructed by operating on a complete set of functions.

The construction of Foldy-Wouthuysen operators $F_\Lambda(\vec{p})$ from the Lorentz transformation out of the rest system can be continued for all higher spins using the same formal procedure.

To find the explicit form of $F_\Lambda^{-1}(\vec{p})$ for arbitrary spin, one needs the form of $H/|H|$. The procedure for calculating $H/|H|$ will be given in the next section.

B. Calculation of $\frac{H}{|H|}$ for Arbitrary Spin

To calculate $\frac{H}{|H|}$ for arbitrary spin consider the rest system sign operator equation,

$$\beta \Psi^{\circ}_{\epsilon, m_\Lambda}(x^0) = \epsilon \Psi^{\circ}_{\epsilon, m_\Lambda}(x^0) , \tag{82}$$

and operate from the left with $\Lambda_\Lambda(\epsilon\vec{p})$. This gives

$$\Lambda_\Lambda(\epsilon\vec{p})\, \beta\, \Lambda^{-1}_\Lambda(\epsilon\vec{p})\, \Lambda_\Lambda(\epsilon\vec{p})\, \Psi^{\circ}_{\epsilon, m_\Lambda}(x^0) = \epsilon\, \Lambda_\Lambda(\epsilon\vec{p})\, \Psi^{\circ}_{\epsilon, m_\Lambda}(x^0) . \tag{83}$$

Since $\Lambda_\Lambda(\epsilon\vec{p})\, \Psi^{\circ}_{\epsilon, m_\Lambda}(x^0)$ is a laboratory system eigenfunction, $\Lambda^{-1}_\Lambda(\epsilon\vec{p})$ can be replaced in the equation by $F^{-1}_\Lambda(\vec{p})$. Similarly, $\Lambda_\Lambda(\epsilon\vec{p})$ operates on the rest system eigenfunctions and so can be

replaced by $F_\Lambda(\vec{p})$. The equation then reads

$$F_\Lambda(\vec{p})\,\beta\,F_\Lambda^{-1}(\vec{p})\,\Psi^{(x)}_{\epsilon l m_\Lambda} = \epsilon\,\Psi^{(x)}_{\epsilon l m_\Lambda} . \tag{84}$$

The index ϵ refers to the sign of the energy in all coordinate systems and $\Psi^{(x)}_{\epsilon l m_\Lambda}$ is the laboratory system eigenfunction. Therefore, $F_\Lambda(\vec{p})\,\beta\,F_\Lambda^{-1}(\vec{p})$ is to be interpreted as the laboratory system sign operator for arbitrary spin. That is,

$$F_\Lambda(\vec{p})\,\beta\,F_\Lambda^{-1}(\vec{p}) = H/|H| , \tag{85}$$

for all spins.

For spin one-half, Equation 85 is

$$\frac{[E+m+2\vec{p}\cdot\vec{\alpha}\beta]}{[2m(E+m)]^{\frac{1}{2}}}\,\beta\,\frac{[E+m-2\vec{p}\cdot\vec{\alpha}\frac{H}{|H|}]}{[2m(E+m)]^{\frac{1}{2}}} = \frac{H}{|H|} . \tag{86}$$

Solving for $H/|H|$ gives

$$H/|H| = \frac{2\vec{p}\cdot\vec{\alpha}+m\beta}{E} , \tag{87}$$

the Dirac theory sign operator, which is the expected result for spin

one-half.

For spin one, Equation 85 is

$$\left[\left(\frac{E}{m}-1\right)\left(\hat{\vec{p}}\cdot\vec{\alpha}\right)^2+\frac{\vec{p}\cdot\vec{\alpha}}{m}\beta+1\right]\beta\left[\left(\frac{E}{m}-1\right)\left(\hat{\vec{p}}\cdot\vec{\alpha}\right)^2-\frac{\vec{p}\cdot\vec{\alpha}}{m}\frac{H}{|H|}+1\right]=\frac{H}{|H|}. \tag{88}$$

This is again a matrix equation which can be solved for $H/|H|$. The result is

$$H/|H| = \left[2E\,\vec{p}\cdot\vec{\alpha}+(E^2+p^2)\beta-2\beta(\vec{p}\cdot\vec{\alpha})^2\right]/E^2+p^2, \tag{89}$$

which is the spin one laboratory system sign operator. This operator and the corresponding ones for higher spin have not been known previously. The operator $\frac{H}{|H|}$ is Hermitian and $\left(\frac{H}{|H|}\right)^2=1$ as required for the sign operator. The spin one Hamiltonian is therefore

$$H(\vec{p}) = \frac{E}{E^2+p^2}\left[2E\,\vec{p}\cdot\vec{\alpha}+(E^2+p^2)\beta-2\beta(\vec{p}\cdot\vec{\alpha})^2\right]. \tag{90}$$

Note that the laboratory functions $\Psi^{(x)}_{\epsilon i m a}$ are the unnormalized eigenfunctions of $H(\vec{p})$. They are orthogonal for different values of ϵ since $H(\vec{p})$ is a Hermitian operator. When $H/|H|$ is substituted in $F_1^{-1}(\vec{p})$, the result is

$$F_1^{-1}(\vec{p}) = \left[(mE - E^2 - p^2)(\hat{\vec{p}}\cdot\vec{\alpha})^2 - m\vec{p}\cdot\vec{\alpha}\beta + E^2 + p^2\right] / E^2 + p^2. \quad (91)$$

For spin one, a unitary operator similar to S_{FW} cannot be constructed.

One calculates $H/|H|$ for arbitrary spin in the same way as for spin one-half and spin one.

C. Calculation of the Laboratory System Values of Other Operators

Once the operators $F_\Lambda(\vec{p})$ and $F_\Lambda^{-1}(\vec{p})$ are constructed, the laboratory system values of other operators of interest can be calculated. In particular, the laboratory system polarization operator can be calculated for arbitrary spin. The rest system polarization operator is, from Equation 14b,

$$\vec{\mathcal{O}}_1^{\circ}\cdot\hat{\vec{e}} = \beta\vec{S}\cdot\hat{\vec{e}}. \quad (92)$$

This operator commutes with the rest system sign operator β, this property will be retained under the similarity transformation $F_\Lambda(\vec{p})$. For spin one-half the calculated laboratory system polarization operator is

$$\vec{\mathcal{O}}_{\frac{1}{2}}(\vec{p})\cdot\hat{\vec{e}} = F_{\frac{1}{2}}(\vec{p})\,\beta\vec{S}\cdot\hat{\vec{e}}\,F_{\frac{1}{2}}^{-1}(\vec{p})$$

$$= 2\beta \vec{S}\cdot\hat{\vec{e}} + 2\vec{S}\cdot\hat{\vec{p}}\left[\frac{2\vec{q}\cdot\vec{p}+m\beta}{E} - \beta\right]\hat{\vec{p}}\cdot\hat{\vec{e}}\ , \qquad (93)$$

as reported by Fradkin and Good (20). This operator is $2\vec{S}\cdot\hat{\vec{p}}$ $H/|H|$ when $\hat{\vec{e}}$ is in the direction of motion and $2\beta\vec{S}\cdot\hat{\vec{e}}_\perp$ when $\hat{\vec{e}}$ is perpendicular to the motion.

For spin one, the calculated laboratory system polarization operator is

$$\vec{\Theta}_1(\vec{p})\cdot\hat{\vec{e}} = F_1(\vec{p})\,\beta\vec{S}\cdot\hat{\vec{e}}\,F_1^{-1}(\vec{p})$$

$$= \frac{\perp}{E^2+p^2}\Big\{-2E^2\left(\frac{E}{m}-1\right)\beta\vec{S}\cdot\hat{\vec{p}}\,\hat{\vec{p}}\cdot\hat{\vec{e}} + 2E\left(\frac{E}{m}-1\right)(\hat{\vec{p}}\cdot\vec{S})^2\gamma_5\vec{p}\cdot\hat{\vec{e}}$$

$$+ (mE - E^2 - p^2)\beta\vec{S}\cdot\hat{\vec{e}}\,(\hat{\vec{p}}\cdot\vec{S})^2 + \left(\frac{E}{m}-1\right)(E^2+p^2)(\hat{\vec{p}}\cdot\vec{S})^2\beta\vec{S}\cdot\hat{\vec{e}}$$

$$-(E^2+p^2)\gamma_5\frac{\vec{p}\cdot\vec{S}}{m}\vec{S}\cdot\hat{\vec{e}} + (E^2+p^2)\beta\vec{S}\cdot\hat{\vec{e}}\Big\}\ . \qquad (94)$$

This operator is not Hermitian. For $\hat{\vec{e}}$ in the direction of motion

$$\vec{\Theta}_1(\vec{p})\cdot\hat{\vec{p}} = \frac{1}{E^2+p^2}\left\{m^2\beta\vec{S} - (2E-m)\gamma_5\vec{p}\cdot\vec{S}\,\vec{S}\right\}\cdot\hat{\vec{p}}\ , \tag{95}$$

whereas for $\hat{\vec{e}}$ perpendicular to the motion

$$\vec{\Theta}_1(\vec{p})\cdot\hat{\vec{e}}_\perp = \frac{1}{E^2+p^2}\Big\{(mE-E^2-p^2)\beta\vec{S}\cdot\hat{\vec{e}}_\perp\,(\hat{\vec{p}}\cdot\vec{S})^2$$

$$+\left(\frac{E}{m}-1\right)(E^2+p^2)(\hat{\vec{p}}\cdot\vec{S})^2\beta\vec{S}\cdot\hat{\vec{e}}_\perp$$

$$-(E^2+p^2)\gamma_5\frac{\vec{p}\cdot\vec{S}}{m}\vec{S}\cdot\hat{\vec{e}}_\perp + (E^2+p^2)\vec{S}\cdot\hat{\vec{e}}_\perp\beta\Big\}\ . \tag{96}$$

The laboratory system values of other operators can be constructed in the same way for all spins. The procedure is, given the rest system operator B°_λ for spin s, the laboratory system operator $B_\lambda(\vec{p})$ is

$$B_\lambda(\vec{p}) = F_\lambda(\vec{p})\,B^\circ_\lambda\,F^{-1}_\lambda(\vec{p})\ . \tag{97}$$

IV. INTEGRALS OF THE MOTION

A. Infinitesimal Generators of the Inhomogeneous Proper Lorentz Group

It is well-known classically that a free particle has several integrals of the motion, that is, certain physical quantities which are constants in time. These quantities are the energy, linear momentum, angular momentum, and the velocity of the center-of-energy (relativistic generalization of the notion of center-of-mass). Quantum mechanically one therefore expects that there exists certain operators, corresponding to these physical quantities, whose expectation values are constants in time and which have the correct transformation properties with respect to the inhomogeneous proper Lorentz group.

The term inhomogeneous proper Lorentz group refers to the group formed by the space and time translations and the continuous Lorentz transformations. This is a ten-parameter group whose infinitesimal generators are:

1. The generators of infinitesimal translations along the three coordinate axes, the momentum operator $\vec{P}$.
2. The generator of an infinitesimal time translation, T (T is the Hamiltonian in the particular representation discussed).
3. The generators of infinitesimal rotations about the three coordinate axes, the angular momentum operator $\vec{J}$.
4. The generators of infinitesimal Lorentz transformations along the three coordinate axes, the center-of-energy operator $\vec{G}$.

The ten operators satisfy the following commutation relations:

$$[P_i, P_j]_- = 0 \,, \tag{98a}$$

$$[P_i, T]_- = 0 \,, \tag{98b}$$

$$[J_i, P_j]_- = i \epsilon_{ijk} P_k \,, \tag{98c}$$

$$[J_i, J_j]_- = i \epsilon_{ijk} J_k \,, \tag{98d}$$

$$[J_i, T]_- = 0 \,, \tag{98e}$$

$$[P_i, G_j]_- = \delta_{ij} T \,, \tag{98f}$$

$$[T, G_i]_- = P_i \,, \tag{98g}$$

$$[J_i, G_j]_- = i \epsilon_{ijk} G_k \,, \tag{98h}$$

$$[G_i, G_j]_- = i \epsilon_{ijk} J_k \,. \tag{98i}$$

These infinitesimal generators are the quantum mechanical operators corresponding to the conserved classical quantities and their expectation values are constants in time as will be discussed below.

In the Foldy representation (12) the infinitesimal generators are

$$\vec{P} = \vec{p} = -i\vec{\nabla}, \tag{99a}$$

$$T = i\beta\omega, \tag{99b}$$

$$\vec{J} = \vec{x} \times \vec{P} + \vec{S}, \tag{99c}$$

$$\vec{G} = \frac{i}{2}[\vec{x}\omega\beta + \omega\beta\vec{x}] - it\vec{P} + i\frac{\vec{P}\times\vec{S}}{1+\omega}\beta. \tag{99d}$$

Denoting any of these by $L\varphi$, the Foldy wave equation, Equation 13, is left invariant under the transformation

$$\varphi(x) \rightarrow \varphi'(x) = (1 + i\,\xi L\varphi)\varphi(x), \tag{100}$$

where ξ is the infinitesimal displacement corresponding to $L\varphi$.

It was noted previously that the functions defined by Equation 25 and 26 are a set of Foldy wave functions. This is true because they satisfy Equation 13, the Foldy wave equation, and because they have the Foldy representation of the infinitesimal generators, Equations 99a to 99d. To see that the latter statement is true it is only necessary to calculate the transformation properties of $\varphi(x)$ from the known properties of $\Psi(\omega)$, the spinor function. The infinitesimal generators of the spinor representation are

$$\vec{P} = \vec{P}\ , \tag{101a}$$

$$T = i\,H, \tag{101b}$$

$$\vec{J} = \vec{X} \times \vec{P} + \vec{S}\ , \tag{101c}$$

$$\vec{G} = i\vec{X}H - it\vec{P} + \vec{a}\ , \tag{101d}$$

where Ψ is assumed to be a scalar with respect to space and time displacements. From these generators and the relation between operators in the spinor representation, L_Ψ , and operators in the Foldy representation, L_φ ,

$$L_\Psi = \omega^{-\frac{1}{2}} F L_\varphi F^{-1} \omega^{\frac{1}{2}}, \tag{102}$$

it is found that $\Psi(x)$ as given in Equation 26 has the infinitesimal generators, Equations 99a to 99d.

B. Integrals of the Motion in Terms of Foldy Wave Functions

Because the integrals have a simple form it is useful to first discuss the integrals of the motion in terms of the Foldy wave functions.

For every transformation

$$X'_\mu = X'_\mu(x) \quad , \quad \varphi' = \varphi'(\varphi) \, , \tag{103}$$

that leaves the form of the Foldy wave equation unchanged there is a conserved quantity

$$\langle D \rangle = \int d\vec{x} \; \varphi^H \beta \, D_\varphi \, \varphi \, , \tag{104a}$$

$$\frac{d}{dt} \langle D \rangle = 0 \, . \tag{104b}$$

In particular for the infinitesimal transformations of Equations 99a to 99d, the conserved quantities are

$$\langle P_i \rangle = \int d\vec{x} \; \varphi^H \beta \, P_i \, \varphi \, , \tag{105a}$$

$$\langle P_4 \rangle = i \int d\vec{x} \; \varphi^H \omega \, \varphi \, , \tag{105b}$$

$$\langle \vec{J} \rangle = \int d\vec{x} \; \varphi^H \beta \left[\vec{x} \times \vec{P} + \vec{S} \right] \varphi \, , \tag{105c}$$

$$\langle \vec{G} \rangle = \int d\vec{x} \; \varphi^H \beta \left\{ \frac{i}{2} \left[\vec{x} \, \omega \beta + \omega \beta \vec{x} \right] - i t \vec{P} + i \frac{\vec{P} \times \vec{S} \beta}{\omega + 1} \right\} \varphi \, . \tag{105d}$$

The quantity $\langle P_\mu \rangle$, the 4-momentum, written in terms of the expansion coefficients is

$$\langle P_i \rangle = \int \frac{d\vec{p}}{E} \sum_{\epsilon, m_s} \epsilon p_i \, |A(\vec{p})_{\epsilon, m_s}|^2 , \tag{106a}$$

$$\langle P_4 \rangle = \int \frac{d\vec{p}}{E} \sum_{\epsilon, m_s} i E \, |A(\vec{p})_{\epsilon, m_s}|^2 . \tag{106b}$$

From the transformation properties of the $A_{\lambda\mu}$, Equations 45, 50, 64 and 65, it is seen that $\langle P_\mu \rangle$ is a 4-vector, regular by space reflection and pseudo by time reflection as required for the energy-momentum.

The tensor $\Theta_{\mu\nu}$ can be constructed from $\langle G_i \rangle$ and $\langle J_k \rangle$ by the definitions

$$\Theta_{ij} = \epsilon_{ijk} \langle J_k \rangle , \tag{107a}$$

$$\Theta_{i4} = -\Theta_{4i} = \langle G_i \rangle , \tag{107b}$$

$$\Theta_{44} = 0 . \tag{107c}$$

With respect to continuous Lorentz transformations, $\Theta_{\mu\nu}$ is a second-rank tensor. The proof of this is given in Appendix IV as part of a discussion of the tensor properties of covariantly defined operators. With respect to the space reflection, Equation 51a,

$$\langle J_i \rangle' = \langle J_i \rangle , \tag{108a}$$

$$\langle G_i \rangle' = -\langle G_i \rangle, \tag{108b}$$

and with respect to the time reflection, Equation 51b,

$$\langle J_i \rangle' = -\langle J_i \rangle, \tag{109a}$$

$$\langle G_i \rangle' = \langle G_i \rangle, \tag{109b}$$

so that $\Theta_{\mu\nu}$ is regular with respect to the space reflection and pseudo with respect to the time reflection.

If D is the identity transformation, one gets the quantity

$$Q = e \int d\vec{x}\, \Phi^{\dagger} \beta \Phi$$

$$= e \int \frac{d\vec{p}}{E} \sum_{\sigma, m_s} \epsilon \left| A^{(\vec{p})}_{\sigma, m_s} \right|^2, \tag{110}$$

where e is the positronic charge. This integral, Q, is the charge for the field. It transforms like a scalar with respect to continuous Lorentz transformations and time reflections and is pseudoscalar with respect to space reflections. Also it is seen from Equation 63 that changes sign under charge conjugation.

The quantity,

$$N = \int d\vec{x}\, \varphi^{\dagger} \varphi$$

$$= \int \frac{d\vec{p}}{E} \sum_{\sigma, m_s} |A(\vec{p})_{\sigma m_s}|^2, \tag{111}$$

is to be interpreted as the number of particles. It is a constant in time and a scalar for both the continuous Lorentz transformations and the reflections. For a Fermi particle N must be normalized to unity.

C. Integrals of the Motion in Terms of Spinors

The discussion of the integrals of the motion can also be made in terms of the spinor functions $\Psi(x)$. From Equation 25, the relation between φ and Ψ, it is found that

$$\langle D \rangle = \frac{1}{m^{2A}} \int d\vec{x}\, \Psi^{\dagger} \omega F^{-1\,\dagger} F^{-1} \frac{H}{|H|} D_{\Psi} \Psi, \tag{112}$$

where $H/|H|$ is the laboratory system sign operator for arbitrary spin as discussed in Chapter III. The connection between D_{Ψ} and D_{φ} for arbitrary spin is

$$D_{\Psi} = \omega^{-\frac{1}{2}} F D_{\varphi} F^{-1} \omega^{\frac{1}{2}}. \tag{113}$$

The integrals of the motion for spin zero, spin one-half, and spin one will be discussed as examples.

1. Spin zero integrals

For the case of spin zero, $F_0(\vec{p}) = 1$ so

$$\langle D_0 \rangle = \int d\vec{x}\, \Psi^{\dagger} \omega \beta\, D_{0\Psi}\, \Psi, \tag{114}$$

where

$$D_{0\Psi} = \omega^{-\frac{1}{2}} D_{0\phi}\, \omega^{\frac{1}{2}}. \tag{115}$$

Equation 114 must be connected to the usual invariant integral, I, appropriate for scalar solutions of the Klein-Gordon equation. To show the connection, one starts with I, as follows. For two solutions ϕ_1, ϕ_2 of the Klein-Gordon equation, the invariant integral is

$$\begin{aligned} I &= i \int d\vec{x} \left(\phi_1^{c} \frac{\partial \phi_2}{\partial t} - \frac{\partial \phi_1^{c}}{\partial t} \phi_2 \right) \\ &= \int d\vec{x} \left(i\phi_1^{c}, \frac{\partial \phi_1^{c}}{\partial t} \right) \beta \begin{pmatrix} -i\phi_2 \\ \frac{\partial \phi_2}{\partial t} \end{pmatrix} \\ &= \int d\vec{x} \left(i\omega^{\frac{1}{2}} \phi_1^{c}, \omega^{-\frac{1}{2}} \frac{\partial \phi_1^{c}}{\partial t} \right) \beta \begin{pmatrix} -i\omega^{\frac{1}{2}} \phi_2 \\ \omega^{-\frac{1}{2}} \frac{\partial \phi_2}{\partial t} \end{pmatrix}, \end{aligned} \tag{116}$$

where in the last equation the Hermitian operators $\omega^{\pm\frac{1}{2}}$ have been inserted. The Foldy wave function Φ_i is related to the solution of the Klein-Gordon equation ϕ_i by

$$\Phi_i = \begin{pmatrix} -i\omega^{\frac{1}{2}}\phi_i \\ \omega^{-\frac{1}{2}}\frac{\partial\phi_i}{\partial t} \end{pmatrix}, \tag{117}$$

thus,

$$I = \int d\vec{x}\, \Phi_1^{\dagger} \beta \Phi_2 . \tag{118}$$

For spin zero Φ and Ψ are related by

$$\Psi_i = \omega^{-\frac{1}{2}} \Phi_i$$

$$= \begin{pmatrix} -i\phi_i \\ \frac{1}{\omega}\frac{\partial\phi_i}{\partial t} \end{pmatrix}$$

$$= -i \begin{pmatrix} \phi_i \\ \phi_i \end{pmatrix}, \tag{119}$$

so that Ψ_i and ϕ_i are identical up to a phase factor $e^{-i\pi/2}$. From Equations 118 and 119 the invariant integral is then

$$I = \int d\vec{x}\, \Psi_1^{\dagger} \omega \beta \Psi_2 . \tag{120}$$

With the identifications $\Psi_2 = D_0 \Psi$ and $\Psi_1 = \Psi$ it is seen that $I = \langle D_0 \rangle$.

2. Spin one-half integrals

The spin one-half operator $F_{\frac{1}{2}}(\vec{p})$ is given by Equation 72. Direct calculation shows that

$$F_{\frac{1}{2}}^{-1\,H}(\vec{p})\, F_{\frac{1}{2}}^{-1}(\vec{p}) = \frac{m}{\omega}\,, \tag{121}$$

and

$$\langle D_{\frac{1}{2}} \rangle = \int d\vec{x}\; \Psi^{H} \frac{H}{|H|} D_{\frac{1}{2}} \Psi\,, \tag{122}$$

where

$$D_{\frac{1}{2}}\Psi = \omega^{-\frac{1}{2}} F_{\frac{1}{2}}\, D_{\frac{1}{2}} F_{\frac{1}{2}}^{-1}\, \omega^{\frac{1}{2}}\,. \tag{123}$$

From Equations 71 and 76 it is seen that

$$D_{\frac{1}{2}}\Psi = S_{F\omega}^{-1}\, D_{\frac{1}{2}}\, S_{F\omega}\,, \tag{124}$$

that is $D_{\frac{1}{2}}$ are the Foldy-Wouthuysen transforms of the Dirac theory invariant integral since

$$\Psi_{\frac{1}{2}} = \left(\frac{m}{\omega}\right)^{\frac{1}{2}} \Lambda_{\frac{1}{2}}(x)\, \varphi_{\frac{1}{2}}$$

$$= S_{F\omega}^{-1}\, \varphi_{\frac{1}{2}} \tag{125}$$

are the Dirac theory laboratory system wave functions.

3. Spin one integrals

Equation 91 gives the spin one operator F_1^{-1} and from it one finds that

$$F_1^{-1\,H} F_1^{-1} = 1 - \frac{2(\vec{P}\cdot\vec{\alpha})^2}{p^2+\omega^2} \quad . \tag{126}$$

Substituting in Equation 112 it is found that

$$\langle D_1 \rangle = \int d\vec{x}\, \Psi^{H} \frac{\omega}{m^2} \left\{ 1 - \frac{2(\vec{P}\cdot\vec{\alpha})^2}{p^2+\omega^2} \right\} \frac{H}{|H|} D_\Psi \Psi \, . \tag{127}$$

In the same way the integrals of the motion for arbitrary spin can be constructed from the displacement operators D_Ψ and the spinor functions Ψ .

V. MASSLESS LIMIT OF THE THEORY

A. Massless Limit of the Spinors

To discuss the massless limit the polarization direction $\hat{\epsilon}$ is chosen, for simplicity, to be in the $\hat{p}$ direction. With this choice the laboratory system spinor function $\Psi(x)$ as given in Equation 24 becomes

$$\Psi(x) = \frac{m_a}{(2\pi)^{3/2}} \int \frac{d\vec{p}}{E} \sum_{\epsilon, m_s} A(\vec{p}) \Lambda(\epsilon \vec{p}) \psi^{\circ}_{\epsilon, m_s}(\hat{p}) e^{i(\vec{p}\cdot\vec{x} - \epsilon E t)} . \tag{128}$$

From Equations 22a and 23a it is seen that

$$\Lambda(\epsilon \vec{p}) = e^{-\operatorname{arctanh}(p/E)\, \epsilon \gamma_5 \hat{p}\cdot\vec{S}} , \tag{129}$$

and from Equations 14a and 14b that

$$\vec{S}\cdot\hat{p}\, \psi^{\circ}_{\epsilon, m_s}(\hat{p}) = \epsilon m_s \psi^{\circ}_{\epsilon, m_s}(\hat{p}) . \tag{130}$$

Since γ_5 and $\vec{S}$ commute it is then true that

$$\Lambda(\epsilon \vec{p})\, \psi^{\circ}_{\epsilon, m_s}(\hat{p}) = e^{-m_s \gamma_5 \operatorname{arctanh} p/E}\, \psi^{\circ}_{\epsilon, m_s}(\hat{p}) , \tag{131}$$

where the exponential on the right is a polynomial in $-\gamma_5 m_\Lambda$ of the same degree and with the same coefficients as the polynomial form of $\Lambda(\epsilon\vec{p})$. Now

$$e^{-m_\Lambda \gamma_5 \operatorname{arctanh} p/\epsilon} = \begin{pmatrix} e^{m_\Lambda \operatorname{arctanh} p/\epsilon} & 0 \\ 0 & e^{-m_\Lambda \operatorname{arctanh} p/\epsilon} \end{pmatrix}. \tag{132}$$

If m_Λ is an integer, say $m_\Lambda = \pm n$ where n is positive, $n = 0, 1, 2, \ldots, \Lambda$ then

$$e^{-m_\Lambda \gamma_5 \operatorname{arctanh} p/\epsilon} = \begin{pmatrix} e^{\pm n \operatorname{arctanh} p/\epsilon} & 0 \\ 0 & e^{\mp n \operatorname{arctanh} p/\epsilon} \end{pmatrix}$$

$$= \begin{pmatrix} [\cosh \operatorname{arctanh} p/\epsilon \pm \sinh \operatorname{arctanh} p/\epsilon]^n & 0 \\ 0 & [\cosh \operatorname{arctanh} p/\epsilon \mp \sinh \operatorname{arctanh} p/\epsilon]^n \end{pmatrix}. \tag{133}$$

Since

$$\cosh \operatorname{arctanh} p/E = E/m\,, \tag{134a}$$

$$\sinh \operatorname{arctanh} p/E = p/m\,, \tag{134b}$$

$$e^{\mp n \gamma_5 \operatorname{arctanh} p/E} = \begin{pmatrix} \left(\frac{E}{m} \pm \frac{p}{m}\right)^n & 0 \\ 0 & \left(\frac{E}{m} \mp \frac{p}{m}\right)^n \end{pmatrix}$$

$$= m^{-n} \begin{pmatrix} (E \pm p)^n & 0 \\ 0 & (E \mp p)^n \end{pmatrix}. \tag{135}$$

Similarly if m_λ is a half-integer say $m_\lambda = \pm k/2$ where $k = 1, 2, 3, \cdots, 2\lambda$ and k positive, then

$$e^{\mp \frac{k}{2} \gamma_5 \operatorname{arctanh} \frac{p}{E}} = \begin{pmatrix} \left[\cosh \tfrac{1}{2} \operatorname{arctanh} \frac{p}{E} \pm \sinh \tfrac{1}{2} \operatorname{arctanh} \frac{p}{E}\right]^k & 0 \\ 0 & \left[\cosh \tfrac{1}{2} \operatorname{arctanh} \frac{p}{E} \mp \sinh \tfrac{1}{2} \operatorname{arctanh} \frac{p}{E}\right]^k \end{pmatrix}, \tag{136}$$

and

$$\cosh \tfrac{1}{2} \operatorname{arctanh} p/E = \left(\frac{E+m}{2m}\right)^{\frac{1}{2}}, \tag{137a}$$

$$\sinh \tfrac{1}{2} \operatorname{arctanh} p/E = \left(\frac{E-m}{2m}\right)^{\frac{1}{2}}. \tag{137b}$$

Thus

$$e^{\mp \frac{k}{2} \gamma_5 \operatorname{arctanh} \frac{p}{E}} = \begin{pmatrix} \left[\left(\frac{E+m}{2m}\right)^{\frac{1}{2}} \pm \left(\frac{E-m}{2m}\right)^{\frac{1}{2}}\right]^k, & 0 \\ 0, & \left[\left(\frac{E+m}{2m}\right)^{\frac{1}{2}} \mp \left(\frac{E-m}{2m}\right)^{\frac{1}{2}}\right]^k \end{pmatrix}$$

$$= m^{-\frac{k}{2}} \begin{pmatrix} \left[\left(\frac{E+m}{2}\right)^{\frac{1}{2}} \pm \left(\frac{E-m}{2}\right)^{\frac{1}{2}}\right]^k, & 0 \\ 0, & \left[\left(\frac{E+m}{2}\right)^{\frac{1}{2}} \mp \left(\frac{E-m}{2}\right)^{\frac{1}{2}}\right]^k \end{pmatrix}. \tag{138}$$

From Equations 135 and 138 it is therefore easy to see that

$$\lim_{m \to 0} m^{\lambda} e^{-m_{\lambda} \gamma_5 \operatorname{arctanh} p/E} = 0 \tag{139}$$

unless $m_\Lambda = \pm\Lambda$. This means that only the polarization directions with and against the momentum remain in $\Psi(x)$ in the massless limit. One also sees from Equations 135 and 138 that in the massless limit only the matrix element with the $+$ sign contributes, so

$$\lim_{m\to 0} m^\Lambda e^{-\Lambda\gamma_5 \arctan\frac{p}{E}} = \begin{pmatrix} (2p)^\Lambda & 0 \\ 0 & 0 \end{pmatrix}, \tag{140a}$$

and

$$\lim_{m\to 0} m^\Lambda e^{\Lambda\gamma_5 \arctan\frac{p}{E}} = \begin{pmatrix} 0 & 0 \\ 0 & (2p)^\Lambda \end{pmatrix}. \tag{140b}$$

Thus the spinor function $\Psi(x)$ becomes

$$\lim_{m\to 0} \Psi(x) = \frac{2^{\Lambda-\frac{1}{2}}}{(2\pi)^{3/2}} \int d\vec{p}\; p^{\Lambda-1} \sum_{\epsilon} \left[A_{\epsilon,\Lambda}(\vec{p}) \begin{pmatrix} \chi_{\epsilon,\Lambda}(\hat{p}) \\ 0 \end{pmatrix} + A_{\epsilon,-\Lambda}(\vec{p}) \begin{pmatrix} 0 \\ \epsilon\chi_{-\epsilon,\Lambda}(\hat{p}) \end{pmatrix} \right] e^{i(\vec{p}\cdot\vec{x} - \epsilon E t)}. \tag{141}$$

This spinor has $2\times(2\Lambda+1)$ components and is therefore the spin generalization of the four-component neutrino theory. To reduce the number of components to $2\Lambda+1$ for parity nonconserving theories, one invokes the criterion that only the part of Ψ which transforms like a lower-dotted spinor, that is the upper $2\Lambda+1$ components, shall be

kept. Note that this is equivalent to the chirality condition

$$\frac{(1-\gamma_5)}{2}\Psi = \Psi . \tag{142}$$

With this condition Equation 141 becomes

$$\Psi_{m=0}(x) = \frac{2^{\lambda-\frac{1}{2}}}{(2\pi)^{3/2}}\int d\vec{p}\, p^{\lambda-1}\left[A_+ u_\lambda e^{i(\vec{p}\cdot\vec{x}-pt)} + A_- \underline{u}_\lambda e^{i(\vec{p}\cdot\vec{x}+pt)}\right], \tag{143}$$

where $A_+ \equiv A_{+,\lambda}$ and $A_- \equiv A_{-,\lambda}$. This spinor is $2^{\lambda-\frac{1}{2}}$ times the massless particle spinor discussed by Hammer and Good (16).

For spin one-half, Equation 143 is the two-component neutrino spinor and for spin one is $\sqrt{2}$ times the three-component photon spinor, as discussed in detail by Good (22).

This procedure for taking the massless limit is especially interesting because it shows that massless particles can be built by a continuous process from massive particles.

B. Massless Limit of Integrals of the Motion

One requires that the integrals of the motion $\langle D \rangle$ as given by Equation 112 reduce to the massless particle theory integrals of the motion when the limit $m \to 0$ is taken.

Consider the various quantities in $\langle D \rangle$,

$$\lim_{m\to 0} \Psi = \lambda^{\Delta-\frac{1}{2}} \bar{\Psi} \,, \tag{144}$$

where $\bar{\Psi}$ is the massless spinor of Hammer and Good. Also

$$\lim_{m\to 0} \frac{\omega}{m^{2\Delta}} F^{-1H} F^{-1} \Psi = \frac{P^{-2\Delta+1}}{2^{\Delta-\frac{1}{2}}} \bar{\Psi} \,, \tag{145}$$

so

$$\langle \bar{D} \rangle = \lim_{m\to 0} \langle D \rangle = \int d\vec{x}\, \bar{\Psi}^{H} P^{-2\Delta+1} \frac{\bar{H}}{|H|} \bar{D}_{\Psi} \bar{\Psi} \,, \tag{146}$$

where $\bar{H}/|H|$ and $\bar{D}_{\Psi}$ are the massless limits of $H/|H|$ and D_{Ψ} respectively. The explicit forms of $\bar{D}_{\Psi}$ and $\bar{H}/|H|$ are found by setting the mass equal to zero in the formulas with mass and remembering that the operators are operating on the functions $U_{\pm}(\hat{p})$. It was found by Hammer and Good (16) that the functions $\bar{\varphi}$ defined by

$$\bar{\Psi} = P^{\Delta-\frac{1}{2}} \bar{\varphi}$$

$$= |\bar{H}|^{\Delta-\frac{1}{2}} \bar{\varphi} \,, \tag{147}$$

are useful for writing the integrals of the motion $\langle \bar{D} \rangle$. As can be seen from Equation 143,

$$\bar{\varphi} = \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{p}}{\sqrt{p}} \left[A_+ u_\Delta e^{i(\vec{p}\cdot\vec{x} - pt)} + A_- u_{-\Delta} e^{i(\vec{p}\cdot\vec{x} + pt)} \right]. \tag{148}$$

Substituting Equation 147 into Equation 146 one gets

$$\langle \bar{D} \rangle = \int d\vec{x}\, \bar{\varphi}^H \frac{1}{p^{\Delta - \frac{1}{2}}} \frac{\bar{H}}{|H|} \bar{D}_\Psi\, p^{\Delta - \frac{1}{2}} \bar{\varphi}$$

$$= \int d\vec{x}\, \bar{\varphi}^H \frac{\bar{H}}{|H|} \bar{D}_\varphi \bar{\varphi}\,, \tag{149}$$

where

$$\bar{D}_\varphi = |\bar{H}|^{-\Delta + \frac{1}{2}} \bar{D}_\Psi |\bar{H}|^{\Delta - \frac{1}{2}}. \tag{150}$$

C. Massless Limit of Integrals of the Motion as a Double Integral on Coordinates

It has been shown by Good (22) that the integrals of the motion for

the photon may be written as a certain type of double integral involving the spinors and the displacement operators of the corresponding transformations. The purpose of this section is to write the integrals of the motion for arbitrary spin in the form of a double integral on the coordinates so that the spin one specialization of the massless limit is the photon double integral.

The solution of the differential equation

$$(m^2 - \nabla^2)\, G(\vec{x}) = \delta(\vec{x}) \tag{151}$$

with the boundary condition $G \to 0$ as $|\vec{x}| \to \infty$ where $\delta(\vec{x})$ is the Dirac delta function, is

$$G(\vec{x}) = \frac{e^{-m|\vec{x}|}}{4\pi|\vec{x}|} \,. \tag{152}$$

The function $G(\vec{x})$ is the Yukawa potential for $m \neq 0$ and the Coulomb potential for $m = 0$. Equation 112 for $\langle D \rangle$ may be rewritten as a double integral by inserting the delta function as defined by Equation 151. Thus,

$$\langle D \rangle = \int d\vec{x} \int d\vec{y} \left(\frac{\omega(y)}{m^{2\Delta}} F^{-1H}(y) F^{-1}(y) \Psi(y) \right)^{H} \left(\frac{H}{|H|} D_{\mathrm{I}} \Psi(x) \right) \times (m^2 - \nabla_x^2) \frac{e^{-m|\vec{x}-\vec{y}|}}{4\pi|\vec{x}-\vec{y}|} \,. \tag{153}$$

By a partial integration on X discarding the surface terms, one gets

$$\langle D\rangle = \int d\vec{x}\int d\vec{y}\left(\frac{\omega(y)}{m^{2\Delta}}F^{-1\,H}_{(y)}F^{-1}_{(y)}\Phi_{(y)}\right)^{H}\frac{e^{-m|\vec{x}-\vec{y}|}}{4\pi|\vec{x}-\vec{y}|}(m^2-\nabla_x^2)\frac{H}{|H|}D_\Phi\Phi(x)\,. \tag{154}$$

Since $\frac{H}{|H|}D_\Phi\Phi$ is a laboratory system wave function, one can replace $m^2-\nabla_x^2$ by $|H(x)|^2$. In the massless limit, Equation 154 becomes

$$\langle\bar{D}\rangle = \int d\vec{x}\int d\vec{y}\left(\frac{1}{p^{2\Delta-1}}\bar{\Phi}_{(y)}\right)^{H}\frac{1}{4\pi|\vec{x}-\vec{y}|}|\bar{H}|^2\frac{\bar{H}}{|H|}\bar{D}_\Phi\bar{\Phi}(x)\,. \tag{155}$$

By two partial integrations, again discarding surface terms, Equation 155 becomes

$$\langle\bar{D}\rangle = \frac{1}{4\pi}\int d\vec{x}\int d\vec{y}\,\frac{\bar{\Phi}(y)}{|\vec{x}-\vec{y}|}\,|\bar{H}|^{-2\Delta+2}\,\bar{H}(x)\,\bar{D}_\Phi\bar{\Phi}(x)\,. \tag{156}$$

For spin one this is

$$\langle\bar{D}_1\rangle = \frac{1}{4\pi}\int d\vec{x}\int d\vec{y}\,\frac{\bar{\Phi}(y)}{|\vec{x}-\vec{y}|}\,\bar{H}(x)\,\bar{D}_\Phi\,\bar{\Phi}(x)\,, \tag{157}$$

which is the double integral of Good (22).

VI. CONNECTION WITH OTHER THEORIES

A. Spin Zero

The spin zero specialization of the arbitrary spin theory, as discussed in Chapter IV, is the usual Klein-Gordon theory in a two-component Hamiltonian form. It was shown in Equation 119 that the spinor function $\Psi(x)$, which satisfies the wave equation

$$\omega\beta\Psi = i\frac{\partial\Psi}{\partial t}, \tag{158}$$

and the Klein-Gordon function Φ are equal up to a phase factor. The connection between the Klein-Gordon invariant integral and the integrals of the motion in the spinor theory has also been discussed.

The laboratory system Hamiltonian, $\omega\beta$ presented in this work is different from the Hamiltonian discussed by Taketani and Sakata (10). They presented a Hamiltonian theory of spin zero and spin one by rewriting the Klein-Gordon and Proca equations in a Hamiltonian form. Their theory has the equations

$$H\Psi = i\frac{\partial\Psi}{\partial t}, \tag{159a}$$

where

$$H = \sigma_3\left[m + \frac{p^2}{2m}\right] + i\sigma_1\left[\frac{p^2}{2m} - \frac{(\vec{\Delta}\cdot\vec{p})^2}{m}\right], \tag{159b}$$

and

$$\vec{p} = -i\vec{\nabla} . \tag{159c}$$

Here $\sigma_1, \sigma_2, \sigma_3$ are a set of $2x2$ Pauli matrices and $\vec{\lambda}$ are the spin matrices for spin zero and spin one. For spin zero the $\vec{\lambda}$ are zero. To make a physical interpretation of the theory by constructing a current and density from Equations 159a and 159b, it is necessary to define the expectation of an operator O by

$$\langle O \rangle = \int \psi^c \sigma_3 O \psi \, d\vec{x} , \tag{160}$$

so that σ_3 is the metric (indefinite). The total charge Q described by the wave function ψ is

$$Q = e \int \psi^c \sigma_3 \psi \, d\vec{x} . \tag{161}$$

Note that although the metric is the same in the Taketani-Sakata theory as in the spinor theory, the interpretation is different. In the spinor theory, $\beta = \sigma_3$ is the sign operator whereas in the Taketani-Sakata theory, it is not.

The Taketani-Sakata theory has been transformed to the Foldy representation by Case (11) with the operator

$$U = e^{\frac{1}{2} i \sigma_2 \theta} , \tag{162}$$

where Θ is defined by the operator equation

$$\Theta = -\arctan i \frac{\left[\frac{p^2}{2m} - \frac{(\vec{\lambda}\cdot\vec{p})^2}{m}\right]}{m + p^2/2m}, \tag{163}$$

so that

$$H_{FW} = \beta W = U^{-1} H U. \tag{164}$$

B. Spin One-Half

The spin one-half specialization of the arbitrary spin theory is the Dirac theory as previously discussed in Chapter IV, Equations 121 through 125. The massless limit of the theory is the four-component neutrino. Application of the chirality condition, Equation 142, reduces the massless theory to the two-component neutrino.

C. Spin One

The usual theory for particles with spin one and mass is the Proca theory which has the field equations

$$\left(\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\mu} - m^2\right) A_\nu = 0, \tag{165a}$$

$$\frac{\partial}{\partial x_\mu} A_\mu = 0, \tag{165b}$$

where A_M is a four-vector. The field equations can be written in the Foldy representation in the following way (following Foldy (12)).

$$\frac{\partial A_i}{\partial t} = -E_i + \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_R} E_R \,, \tag{166a}$$

$$\frac{\partial E_i}{\partial t} = A_i + \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_m} A_m - \frac{\partial}{\partial x_\ell}\frac{\partial}{\partial x_\ell} A_i \,, \tag{166b}$$

where

$$E_i \equiv i\frac{\partial}{\partial x_i} A_4 - i\frac{\partial}{\partial x_4} A_i \,. \tag{166c}$$

With the further definition

$$\theta_{ij} \equiv \partial \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} - \delta_{ij}\frac{\partial}{\partial x_R}\frac{\partial}{\partial x_R} \,, \tag{167a}$$

and the property

$$\theta_{ij}\theta_{jk} = (\omega+m)^2(\omega-m)^2\delta_{ik} \,, \tag{167b}$$

Equations 166a and 166b can be rewritten as

$$\frac{\partial A_i}{\partial t} = \left\{-\tfrac{1}{2}[m^2+\omega^2]\delta_{ij} + \tfrac{1}{2}\theta_{ij}\right\} E_j \,, \tag{168a}$$

$$\frac{\partial E_i}{\partial t} = \left\{ \tfrac{1}{2}\left[m^2+\omega^2\right]\delta_{ij} + \tfrac{1}{2} g_{ij} \right\} A_j \,. \qquad (168b)$$

Making the identifications

$$U_i = \frac{i}{(2\omega)^{\frac{1}{2}}} \left[(\omega+m)\delta_{ij} + \frac{g_{ij}}{\omega+m} \right] A_j \,, \qquad (169a)$$

and

$$V_i = \frac{1}{(2\omega)^{\frac{1}{2}}} \left[(\omega+m)\delta_{ij} - \frac{g_{ij}}{\omega+m} \right] \frac{E_j}{m} \,, \qquad (169b)$$

the function

$$\Phi = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ V_1 \\ V_2 \\ V_3 \end{pmatrix} \qquad (170)$$

is the Foldy wave function for spin one satisfying Equation 13. The relation between Φ and the spinor function Ψ for spin one is

$$\Psi(x) = m\omega^{-\frac{1}{2}} \Lambda_1(x)\, \varphi(x)\,, \tag{171}$$

where $\Lambda_1(x)$ is given by Equation 77 with $\vec{P}$ replaced by $-i\vec{\nabla}$ and ϵ by $P_0/|P_0|$.

VII. SUMMARY AND POSSIBLE EXTENSIONS

A. Summary

The theory presented in this thesis is a C- number theory for a single particle with mass m and arbitrary spin λ.

The starting point of the theory is that all particles with mass have a unique rest system. In this rest system the differential equations describing the time development of the particle are greatly simplified. To get a complete set of rest system states, one takes the eigenfunctions of the rest system polarization operator which commutes with the rest system energy operator. The rest system eigenstates are then specified by the sign of the energy and the projection of the spin.

To construct the theory in the laboratory system, one assumes that the functions are symmetric spinors and applies the Lorentz transformation rules. That the functions are indeed spinors is demonstrated by the successful proofs of covariance.

As discussed in Chapter III, the laboratory system Hamiltonians analogous to the spin one-half Hamiltonian can be calculated, in principle, for all spins. This is made possible by the extension of the notion of the Foldy-Wouthuysen transformation as the Lorentz transformation out of the rest system to arbitrary spin. As well as the Hamiltonian, the laboratory system values of other interesting operators, such as the polarization operator, can be calculated.

In addition, the theory provides the plane wave solutions of the laboratory system wave equation as well as the transformation properties

of the solutions with respect to the inhomogeneous proper Lorentz group, the reflections, and charge conjugation.

The massless limit of the arbitrary spin theory (including the chirality condition) is the $(2A+1)$- component spinor theory of Hammer and Good (16). This theory contains the two-component neutrino and the photon as well as all higher spins. In connection with the massless limit it was found possible to write the integrals of the motion, for arbitrary spin, as a double integral on coordinates of the spinors and displacement operators, the spin one specialization being the double integral for the photon discussed by Good (22).

B. Possible Extensions

1. Interactions

The theory presented in this thesis has provided the laboratory Hamiltonians appropriate for discussing all spins. For spin one this Hamiltonian is different from the Taketani-Sakata Hamiltonian, and it is possible that the form of interactions in this theory will differ from the interactions proposed for other theories. If such is the case, an experimental test may be possible for the form of spin one interactions.

2. Second quantization

This theory is a single particle theory and as such is subject to the usual limitations of such theories. The Hamiltonian has negative as well as positive energy eigenvalues, and also the theory does not allow for the creation and destruction of particles in reactions. Therefore, to get a correct physical interpretation it is necessary to quantize the

theory. The quantization will be easily made because the spinor function $\Psi(x)$ is already expanded in a set of single particle functions. Thus a complete q- number theory can be constructed. Since all spins are treated the same, it will be interesting to see if the connection between the spin and statistics of the particles parallels that of the massless particles of Hammer and Good (23).

Guessing interactions is also important here in the q- number theories. It will be especially interesting to find the interaction between Dirac particles and the spin one meson.

VIII. APPENDIX I: TWO THEOREMS ABOUT LORENTZ TRANSFORMATION MATRICES

In this appendix two theorems used in the proofs of covariance in Chapter II will be proved.

The first theorem, cited in Equation 38, is that

$$e^{-i\vec{\tau}\cdot\vec{A}}\, s_i\, e^{i\vec{\tau}\cdot\vec{A}} = b_{ij}\, s_j , \tag{172}$$

where b_{ij} is the rotation matrix given in Equation 31b. The theorem is true for spin one-half. This can be seen by expanding the exponentials and multiplying the matrices together

$$e^{-i\vec{\tau}\cdot\frac{\vec{\sigma}}{2}}\, \sigma_i\, e^{i\vec{\tau}\cdot\frac{\vec{\sigma}}{2}} = \left[\cos\tfrac{\tau}{2} - i\hat{\tau}\cdot\vec{\sigma}\sin\tfrac{\tau}{2}\right]\sigma_i\left[\cos\tfrac{\tau}{2} + i\hat{\tau}\cdot\vec{\sigma}\sin\tfrac{\tau}{2}\right]$$

$$= \sigma_i\cos^2\tfrac{\tau}{2} - i\,\hat{\tau}\cdot\vec{\sigma}\,\sigma_i\sin\tfrac{\tau}{2} + i\sigma_i\hat{\tau}\cdot\vec{\sigma}\sin\tfrac{\tau}{2}$$

$$+ \hat{\tau}\cdot\vec{\sigma}\,\sigma_i\,\hat{\tau}\cdot\vec{\sigma}\sin^2\tfrac{\tau}{2} . \tag{173}$$

Using the properties

$$\sigma_i \sigma_j = \delta_{ij} + i \epsilon_{ijk} \sigma_k , \tag{174a}$$

and

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2 \delta_{ij} , \tag{174b}$$

the above equation can be simplified to give

$$e^{-i\vec{\tau}\cdot\frac{\vec{\sigma}}{2}} \sigma_i e^{i\vec{\tau}\cdot\frac{\vec{\sigma}}{2}} = \sigma_i \cos\tau + \epsilon_{ijk} \sigma_j \hat{\tau}_k \sin\tau$$

$$+ \hat{\tau}_i \hat{\tau}_j \sigma_j (1 - \cos\tau)$$

$$= b_{ij} \sigma_j . \tag{175}$$

The theorem is then true for all spins by application of another theorem of Hausdorff (18):

$$e^{-A} B e^{A} = B + [B,A]_- + \frac{1}{2!}[[B,A]_-,A]_- + \cdots . \tag{176}$$

In the evaluation of the right hand side of Equation 176, only the commu-

tation relations of the operators are needed and they are the same for all spins. Therefore, Equation 172 is true for all spins.

The second theorem to be proved, Equation 47, is that

$$e^{i\vec{\tau}\cdot\vec{\alpha}}\,\Lambda(\epsilon\vec{p}) = \Lambda(\epsilon\vec{p}')\,e^{i\vec{\tau}_R\cdot\vec{S}}, \tag{177}$$

where $\vec{\tau}$ is given by Equation 30, and $\vec{\tau}_R$ is a pure rotation parameter to be determined. To prove Equation 177, it is convenient to rewrite it as

$$e^{i\vec{\tau}_R\cdot\vec{S}} = \Lambda^{-1}(\epsilon\vec{p}')\,e^{i\vec{\tau}\cdot\vec{\alpha}}\,\Lambda(\epsilon\vec{p}). \tag{178}$$

This equation is true for spin one-half. This can be seen by expanding the exponentials on the right:

$$e^{i\vec{\tau}_R\cdot\vec{S}} = \frac{[E'+m-2\epsilon\vec{p}'\cdot\vec{\alpha}]}{[2m(E'+m)]^{1/2}}\left(\cos\frac{\tau}{2} + 2i\hat{\vec{\tau}}\cdot\vec{\alpha}\sin\frac{\tau}{2}\right)\frac{[E+m+2\vec{p}\cdot\vec{\alpha}]}{[2m(E+m)]^{1/2}}. \tag{179}$$

Substituting for E' and $\vec{p}'$ according to Equations 28a to 28d, the final result is

$$e^{i\vec{\tau}_R\cdot\vec{S}} = \frac{\sqrt{\frac{1+\gamma}{2}}}{\sqrt{\left(\frac{E'}{m}+1\right)\left(\frac{E}{m}+1\right)}}\left\{\frac{E}{m}+1-\frac{\epsilon\gamma\vec{v}\cdot\vec{p}}{(\gamma+1)m}\right\}$$

$$+\frac{i\epsilon_{ijk}\sqrt{\frac{2}{1+\gamma}}\,\gamma\epsilon p_i v_j \sigma_k}{2m\sqrt{\left(\frac{E'}{m}+1\right)\left(\frac{E}{m}+1\right)}}\,. \qquad (180)$$

From Appendix II, one sees that

$$e^{i\vec{\tau}_R\cdot\frac{\vec{\sigma}}{2}} = \cos\frac{\tau_R}{2} + i\vec{\tau}_R\cdot\vec{\sigma}\sin\frac{\tau_R}{2}\,, \qquad (181)$$

so

$$\cos\frac{\tau_R}{2} = \frac{\sqrt{\frac{1+\gamma}{2}}}{\sqrt{\left(\frac{E'}{m}+1\right)\left(\frac{E}{m}+1\right)}}\left\{\frac{E}{m}+1-\frac{\epsilon\gamma\vec{v}\cdot\vec{p}}{m(\gamma+1)}\right\}, \qquad (182a)$$

and

$$\sin\frac{\gamma_R}{2} = \frac{\sqrt{\frac{2}{1+\gamma}}\,\gamma PV}{2m\sqrt{\left(\frac{E'}{m}+1\right)\left(\frac{E}{m}+1\right)}}\sqrt{\left[1-(\hat{\vec{P}}\cdot\hat{\vec{V}})^2\right]}\ , \tag{182b}$$

both coefficients being real.

The axis of the rotation is

$$\hat{\vec{r}}_R = \epsilon\,\widehat{\vec{P}\times\vec{V}}\ . \tag{183}$$

Equation 48 is found from 182a and 182b. Also as can be verified explicitly

$$\cos^2\frac{\gamma_R}{2} + \sin^2\frac{\gamma_R}{2} = 1 \tag{184}$$

Hence the theorem is true for spin one-half. It is then true for all spins by appeal to Equation 35, the first of Hausdorff's theorems.

IX. APPENDIX II: POLYNOMIAL FORM OF THE LORENTZ TRANSFORMATION MATRIX

The purpose of this appendix is to show that $e^{i\vec{\tau}\cdot\vec{J}}$ can be written as a polynomial in $\hat{\vec{\tau}}\cdot\vec{J}$ for arbitrary spin. This can be proved in the following way. Choose $\vec{\tau}$ in the $\hat{X}_3$ direction so that

$$e^{i\vec{\tau}\cdot\vec{J}} = e^{i\tau J_3} . \tag{185}$$

For spin J,

$$J_3 = \begin{pmatrix} J & & & \\ & J-1 & & 0 \\ & & J-2 & \\ & 0 & & \ddots \\ & & & -J \end{pmatrix} . \tag{186}$$

Expanding the exponential gives

$$e^{i\tau J_3} = 1 + \sum_{n=1}^{\infty} \frac{(i\tau)^n}{n!} \begin{pmatrix} J & & \\ & J-1 & 0 \\ & \ddots & \\ 0 & & -J \end{pmatrix}^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(i\tau)^n}{n!} \begin{pmatrix} J^n & & & \\ & (J-1)^n & & 0 \\ & & \ddots & \\ & 0 & & (-J)^n \end{pmatrix} . \tag{187}$$

This equation can be rewritten in terms of trigonometric functions. If s is an integer, Equation 187 is

$$e^{i\tau s_3} = \begin{pmatrix} [\cos\tau+i\sin\tau]^{s} & & & & 0 \\ & [\cos\tau+i\sin\tau]^{s-1} & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \ddots \\ 0 & & & & [\cos\tau+i\sin\tau]^{-(s-1)} \\ & & & & & [\cos\tau+i\sin\tau]^{-s} \end{pmatrix}$$

$$= [\cos\tau+i\sin\tau]^{s} I_{s} + [\cos\tau+i\sin\tau]^{s-1} I_{s-1} + \cdots$$

$$+ [\cos\tau+i\sin\tau] I_{1} + I_{0} + [\cos\tau+i\sin\tau]^{-1} I_{-1} + \cdots$$

$$+ [\cos\tau+i\sin\tau]^{-(s-1)} I_{-(s-1)} + [\cos\tau+i\sin\tau]^{-s} I_{-s} . \qquad (188)$$

If s is a half integer, Equation 187 can be written

$$e^{i\tau J_3} = \begin{pmatrix} \left[\cos\frac{\tau}{2} + i\sin\frac{\tau}{2}\right]^{2J} & & & \\ & \left[\cos\frac{\tau}{2} + i\sin\frac{\tau}{2}\right]^{2(J-1)} & & 0 \\ & & \ddots & \\ & & & \left[\cos\frac{\tau}{2} + i\sin\frac{\tau}{2}\right]^{-2J} \\ & 0 & & \end{pmatrix}$$

$$= \left[\cos\tfrac{\tau}{2} + i\sin\tfrac{\tau}{2}\right]^{2J} I_J + \left[\cos\tfrac{\tau}{2} + i\sin\tfrac{\tau}{2}\right]^{2(J-1)} I_{J-1} + \cdots$$

$$+ \left[\cos\tfrac{\tau}{2} + i\sin\tfrac{\tau}{2}\right] I_{\frac{1}{2}} + \left[\cos\tfrac{\tau}{2} + i\sin\tfrac{\tau}{2}\right]^{-1} I_{-\frac{1}{2}} + \cdots$$

$$+ \left[\cos\tfrac{\tau}{2} + i\sin\tfrac{\tau}{2}\right]^{-2(J-1)} I_{-(J-1)} + \left[\cos\tfrac{\tau}{2} + i\sin\tfrac{\tau}{2}\right]^{-2J} I_{-J}. \qquad (189)$$

The matrices I_{m_J} are a complete set of diagonal matrices for spin J. They have the form

$$I_s = \begin{pmatrix} 1 & & 0 \\ & 0 & \\ 0 & & 0 \end{pmatrix}, \quad I_{s-1} = \begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix}, \ldots$$

$$I_{-(s-1)} = \begin{pmatrix} 0 & & 0 \\ & 0 & \\ 0 & & 1 \end{pmatrix}, \quad I_{-s} = \begin{pmatrix} 0 & & 0 \\ & 0 & \\ 0 & & 0 \end{pmatrix}. \tag{190}$$

The matrices I_{m_s} can be written in terms of powers of s_3 according to

$$I_{m_s} = \prod_{m_s' \neq -m_s} \frac{(s_3 + m_s')}{(m_s + m_s')}. \tag{191}$$

There are $2s$ terms in the product so the set $1, s_3, \ldots, s_3^{2s}$ is also a complete set of diagonal matrices for spin s. Consequently $e^{i\tau s_3}$ can be written as a polynomial in s_3 of degree $2s$. For an arbitrary $\vec{\tau}$ direction, one replaces s_3 by $\hat{\tau}\cdot\vec{s}$ so that $e^{i\vec{\tau}\cdot\vec{s}}$ can be written as a polynomial in $\hat{\tau}\cdot\vec{s}$ of degree $2s$.

So far s a half-integer

$$e^{i\vec{\tau}\cdot\vec{\Lambda}} = \sum_{m_\Lambda=-\Lambda}^{\Lambda} \left[\cos\frac{\tau}{2} + i\sin\frac{\tau}{2}\right]^{2m_\Lambda} \prod_{m'_\Lambda \neq m_\Lambda} \frac{(\hat{\tau}\cdot\vec{\Lambda} + m'_\Lambda)}{(m_\Lambda + m'_\Lambda)} . \tag{192}$$

For example, for $\Lambda = \frac{1}{2}$

$$e^{i\vec{\tau}\cdot\vec{\Lambda}} = \cos\frac{\tau}{2} + 2i\,\hat{\tau}\cdot\vec{\Lambda}\sin\frac{\tau}{2} . \tag{193}$$

For $\Lambda = \frac{3}{2}$,

$$e^{i\vec{\tau}\cdot\vec{\Lambda}} = -\tfrac{4}{3} i \sin^3\tfrac{1}{2}\tau\,(\hat{\tau}\cdot\vec{\Lambda})^3 + 2\left[\cos^3\tfrac{\tau}{2} - \cos\tfrac{\tau}{2}\right](\hat{\tau}\cdot\vec{\Lambda})^2$$

$$+ i\left[2\sin\tfrac{\tau}{2} + \tfrac{1}{3}\sin^3\tfrac{\tau}{2}\right]\hat{\tau}\cdot\vec{\Lambda} + \tfrac{3}{2}\cos\tfrac{\tau}{2} - \tfrac{1}{2}\cos^3\tfrac{\tau}{2} . \tag{194}$$

Similarly if Λ is an integer

$$e^{i\vec{\tau}\cdot\vec{\Lambda}} = \sum_{m_\Lambda=-\Lambda}^{\Lambda} \left[\cos\tau + i\sin\tau\right]^{m_\Lambda} \prod_{m'_\Lambda \neq -m_\Lambda} \frac{(\hat{\tau}\cdot\vec{\Lambda} + m'_\Lambda)}{(m_\Lambda + m'_\Lambda)} . \tag{195}$$

For spin one, this is

$$e^{i\vec{\tau}\cdot\vec{\Lambda}} = (\cos\tau - 1)(\hat{\tau}\cdot\vec{\Lambda})^2 + i\sin\tau\,\hat{\tau}\cdot\vec{\Lambda} + 1. \tag{196}$$

In general to calculate $e^{i\vec{\tau}\cdot\vec{\Lambda}}$ for a given spin, one uses the binomial expansion to calculate $(\cos\tau + i\sin\tau)^{m/s}$ or $(\cos\frac{\tau}{2} + i\sin\frac{\tau}{2})^{2m/s}$. The binomial theorem is

$$(a \pm b)^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j} (\pm b)^j, \tag{197}$$

where $\binom{n}{j} = \frac{n!}{j!(n-j)!}$ and n is an integer.

X. APPENDIX III: CHARGE CONJUGATION MATRIX

In this appendix the choice of the phase in the charge conjugation matrix C, Equation 56, is discussed.

Suppose

$$C = \begin{pmatrix} 0 & u \\ e^{is} u & 0 \end{pmatrix}, \tag{198}$$

where

$$u^c = u, \tag{199a}$$

$$u^{\dagger} = u^{-1}, \tag{199b}$$

and $u^2 = \pm 1$ for bosons/fermions.

The charge conjugate $\bar{\psi}$ of ψ is defined by

$$\bar{\psi} = [C\psi]^c. \tag{200}$$

What is $\bar{\bar{\psi}}$, the charge conjugate of the charge conjugate of ψ ?

$$\bar{\bar{\psi}} = [C\bar{\psi}]^c = [C(C\psi)^c]^c$$

$$= C^c C \psi. \tag{201}$$

So

$$\bar{\Psi} = \begin{pmatrix} e^{i\delta} u^2 & 0 \\ 0 & e^{-i\delta} u^2 \end{pmatrix} \Psi$$

$$= \pm \begin{pmatrix} e^{i\delta} & 0 \\ 0 & e^{-i\delta} \end{pmatrix} \Psi. \tag{202}$$

It is reasonable to require that $\bar{\Psi}$ differ from Ψ by at most an overall phase factor, so the choice

$$e^{i\delta} = e^{-i\delta} \tag{203}$$

must be made. That is,

$$\delta = 0, \pm\pi, \pm 2\pi, \ldots . \tag{204}$$

It is also required of C that

$$C \beta C^{-1} = -\beta . \tag{205}$$

This means that

$$e^{i\delta} = -1, \tag{206}$$

since

$$C\beta C^{-1} = e^{i\delta}\beta \tag{207}$$

from the previous argument. So

$$\delta = \pm\pi, \pm 3\pi, \cdots . \tag{208}$$

The conclusion is that the phase factor $e^{i\delta}$ in C is completely determined within a multiple of 2π as $\delta = \pi$. A result is that

$$\bar{\Psi} = \mp\Psi \tag{209}$$

for bosons/fermions.

A further condition to be imposed on C is that

$$C\Lambda C^{-1} = \Lambda^{c}, \tag{210}$$

where Λ is the spinor Lorentz transformation matrix. This condition comes from the requirement that $\bar{\Psi}$ transform like Ψ with respect to Lorentz transformation. The result of this condition is the requirement

that

$$U \vec{S} U^{-1} = -\vec{S}^{c}. \tag{211}$$

XI. APPENDIX IV: TENSOR PROPERTIES OF OPERATORS

The purpose of this appendix is to discuss the tensor properties of covariantly defined operators. This discussion is pertinent to Chapter IV, Sections A and B. The proofs given in this appendix are due to C. L. Hammer as given in a course in advanced quantum mechanics at Iowa State University, 1962-1963.

The main theorem to be proved is the following. A necessary and sufficient condition that covariantly defined operators $\Theta(x)$ have the proper tensor properties under the pure, continuous transformation

$$\psi'(x) = e^{i\tau\lambda}\,\psi(x), \tag{212a}$$

$$x' = Ax, \tag{212b}$$

is that

$$i\left[\Theta(x)\,\lambda \pm \lambda^{\mu}\,\Theta(x)\right] = a\,\Theta(x) \tag{213}$$

where τ is the displacement, λ is the displacement operator, and "a" is the infinitesimal transformation of a tensor of the same rank as $\Theta(x)$ for the transformation A. By pure transformation is meant rotation, pure Lorentz transformation, and translation. A corollary to the theorem is that a necessary and sufficient condition for studying the con-

tinuous transformation of an operator $\mathcal{O}$ is a study of the infinitesimal transformations of the operator $\mathcal{O}$.

Proof of necessity:

Given Equations 212a, 212b and

$$\langle \mathcal{O}' \rangle_{\alpha\beta\cdots\gamma} = \int d\vec{x}' \, \psi'^{H}(x') \, \mathcal{O}^{(x')}_{\alpha\beta\cdots\gamma} \, \psi'(x') , \tag{214a}$$

$$\langle \mathcal{O} \rangle_{\alpha\beta\cdots\gamma} = \int d\vec{x} \, \psi^{H}(x) \, \mathcal{O}^{(x)}_{\alpha\beta\cdots\gamma} \, \psi(x) . \tag{214b}$$

Substituting for $\psi'(x)$ gives

$$\langle \mathcal{O}' \rangle_{\alpha\beta\cdots\gamma} = \int d\vec{x} \, \psi'^{H}(x) \, \mathcal{O}^{(x)}_{\alpha\beta\cdots\gamma} \, \psi(x)$$

$$= \int d\vec{x} \, \psi^{H}(x) \, (e^{i\tau_\lambda})^{H} \, \mathcal{O}^{(x)}_{\alpha\beta\cdots\gamma} \, (e^{i\tau_\lambda}) \, \psi(x) . \tag{215}$$

Comparing to Equation 214b shows that if $\mathcal{O}(x)$ is to be a tensor,

$$(e^{i\tau_\lambda})^{H} \, \mathcal{O}^{(x)}_{\alpha\beta\cdots\gamma} \, e^{i\tau_\lambda} = a_{\alpha\delta} a_{\beta\sigma} \cdots a_{\gamma\nu} \, \mathcal{O}^{(x)}_{\delta\sigma\cdots\nu} , \tag{216}$$

so that

$$\langle \mathcal{O}' \rangle_{\alpha\beta\cdots\gamma} = a_{\alpha\delta} a_{\beta\sigma} \cdots a_{\gamma\tau} \langle \mathcal{O} \rangle_{\delta\sigma\cdots\tau} . \tag{217}$$

Expanding $e^{i\tau\lambda}$ and $a_{\alpha\delta}$ in Equation 216 gives

$$[1 - i\tau^c \lambda^H] \mathcal{O}^{(x)}_{\alpha\beta\cdots\gamma} [1 + i\tau\lambda] = \{ \delta_{\alpha\delta}\delta_{\beta\sigma}\cdots\delta_{\gamma\tau}$$

$$+ \tau \varepsilon_{\alpha\delta}\delta_{\beta\sigma}\cdots\delta_{\gamma\tau} + \tau\delta_{\alpha\delta}\varepsilon_{\beta\sigma}\cdots\delta_{\gamma\tau} + \cdots$$

$$+ \tau\delta_{\alpha\delta}\delta_{\beta\sigma}\cdots\varepsilon_{\gamma\tau}\} \mathcal{O}^{(x)}_{\delta\sigma\cdots\tau} , \tag{218}$$

where

$$a_{\alpha\delta} = \delta_{\alpha\delta} + \tau\, \varepsilon_{\alpha\delta} , \tag{219a}$$

$$\varepsilon_{\alpha\delta} = \left(\frac{\partial a_{\alpha\delta}}{\partial \tau} \right)_{\tau=0} . \tag{219b}$$

The quantities $\mathcal{E}_{\alpha\delta}$ are called structure constants. In general τ will be more than one parameter. In this event it is understood that the index specifying this is suppressed. Collecting terms gives

$$i\left[\Theta^{(x)}_{\alpha\beta\cdots\gamma}\tau\lambda - \tau^{c}\lambda^{H}\Theta^{(x)}_{\alpha\beta\cdots\gamma}\right] = \tau\left[\mathcal{E}_{\alpha\delta}\delta_{\beta\sigma}\cdots\delta_{\gamma\nu}+\cdots\right.$$

$$\left.+\delta_{\alpha\delta}\cdots\mathcal{E}_{\gamma\nu}\right]\Theta^{(x)}_{\delta\sigma\cdots\nu}, \tag{220}$$

or

$$i\left[\Theta_{(x)}\tau\lambda - \tau^{c}\lambda^{H}\Theta_{(x)}\right] = \tau a\Theta_{(x)}. \tag{221}$$

Since τ is either real or imaginary for pure transformations, $\tau^{c} = \pm\tau$, so that

$$i\left[\Theta\lambda \pm \lambda^{H}\Theta\right] = a\Theta. \tag{222}$$

Note that if τ is real/imaginary and λ is hermitian/antihermitian the above bracket reduces to a commutator. The proof of sufficiency takes several steps and will not be shown here.

This theorem may be applied to the operators of the Foldy represen-

tation since all the conditions are met:

1. For pure rotations τ is real and $\lambda = \lambda^H = \vec{J}_\varphi$. Therefore a covariant operator is a tensor under rotations if

$$[\vec{J}, \Theta]_- = i a \Theta. \tag{223}$$

Thus for tensors up to rank two

Scalars $$[J_i, \Theta]_- = 0, \tag{224a}$$

Vectors $$[J_i, \Theta_j]_- = i \epsilon_{ijk} \Theta_k, \tag{224b}$$

Rank 2 $$[J_i, \Theta_{jk}]_- = i \{ \epsilon_{ijl} \delta_{km} + \epsilon_{ikm} \delta_{jl} \} \Theta_{lm}. \tag{224c}$$

2. The rules for pure Lorentz transformations come from the facts that τ is pure imaginary and $\lambda = -\lambda^H = \vec{G}\varphi$. They give

Scalars $$[G_i, \Theta]_- = 0, \tag{225a}$$

4-vectors $$[G_i, \Theta_k]_- = i \delta_{ik} \Theta_4, \tag{225b}$$

$$[G_i, \Theta_4]_- = -i \Theta_i, \tag{225c}$$

Tensors $$[G_i, \Theta_{jk}]_- = i \{ \delta_{ik} \Theta_{j4} + \delta_{ij} \Theta_{4k} \}, \tag{225d}$$

$$[G_i, \Theta_{j4}]_- = -i \Theta_{ji} + i \delta_{ij} \Theta_{44}, \tag{225e}$$

$$[G_i, \Theta_{44}]_- = -i (\Theta_{i4} + \Theta_{4i}). \tag{225f}$$

The tensor $\Theta_{\mu\nu}$ defined by Equations 107a to 107c satisfies these rules and thus is a second rank tensor with respect to continuous Lorentz transformations.

PART TWO

PHOTOPRODUCTION OF Tm^{173}

XII. INTRODUCTION

In the course of a survey at this Laboratory of the radioactive species produced by synchrotron irradiation of ytterbium, an activity identified as Tm^{173} has been produced. Since Tm^{173} had not been observed prior to this time, an investigation of the nuclear properties was begun. During this investigation Kuroyanagi, et al. (24) reported the production of Tm^{173} by internal bombardment of Yb in the Tohoku University betatron. They found a half-life of 7.2 ± 0.5 hours for Tm^{173}, a beta-ray with end-point 0.9 ± 0.05 Mev, and two gamma rays with energies 400 and 470 kev.

Since Tm^{169} is the only stable thulium isotope, Tm^{173} is expected to decay by negatron emission to excited states of Yb^{173}. Considerable work has recently been done on the energy levels of Yb^{173} (25, 26, 27) with levels found at 78.7, 179.5, 351.2, and 636.8 Kev above the ground state. All of the above levels have been observed by following the electron-capture decay of Lu^{173}. The two lowest levels have also been observed by coulomb excitation (28, 29). Spins and parities of the levels and multipolarities of the gamma ray transitions have been assigned.

The Tm^{173} nucleus is expected to have a ground state spin of one-half like the other odd-A thulium isotopes and hence to decay to low-spin states of Yb^{173} which cannot be reached in the decay of Lu^{173}. To investigate these energy levels Tm^{173} has been produced by synchrotron irradiation of Yb^{174} inside the donut of the Iowa State University synchrotron. This reaction yields low intensities of Tm^{173}.

XIII. EXPERIMENTAL

A. Preparation of Sources

Sources of Tm^{173} were prepared by bombarding enriched ytterbium oxide in the Iowa State University electron synchrotron. The enriched ytterbium, supplied by Oak Ridge National Laboratory, contained the following proportions of the various stable ytterbium isotopes:

Isotope	Enriched %	Natural %
Yb^{168}	0.01	0.14
Yb^{170}	3.03	0.03
Yb^{171}	0.13	14.31
Yb^{172}	0.33	21.82
Yb^{173}	0.78	16.13
Yb^{174}	98.4	31.84
Yb^{176}	0.36	12.73

Samples of ytterbium oxide weighing 80 mg were irradiated. They were contained in a stainless-steel cylinder 7/16-in. long, 1/4 in. o.d., and 1/8 in. i.d. The sample holder was taped to a synthane rod and inserted by means of an internal probe into the donut of the synchrotron. This allowed much higher fluxes in the sample than could be obtained with conventional external bombardment. The synchrotron was operated at a maximum energy of 45 Mev for all the irradiation. After irradiation the samples were removed from the irradiation container and placed in plexiglass holders for counting.

B. Half Life

The half-life of Tm^{173} was determined by following both the β^- decay and the decay of the associated gamma-rays. For the β^- counting

a Los Alamos Sugarman proportional counting tube with a 1.8 mg/cm^2 aluminized Mylar window was used to detect the particles; the counting gas was methane. The output from the counting tube was fed into a Nuclear-Chicago Model 186 P scaler. The counting rate of a Sr^{90} -Y^{90} source was used to normalize the counting data.

The decay of the gamma-rays was followed with a cylindrical 2" x 2" Harshaw NaI(Tl) crystal which was connected to a Nuclear-Chicago Model 1820 recording spectrometer and a Nuclear-Chicago Model 186P scaler.

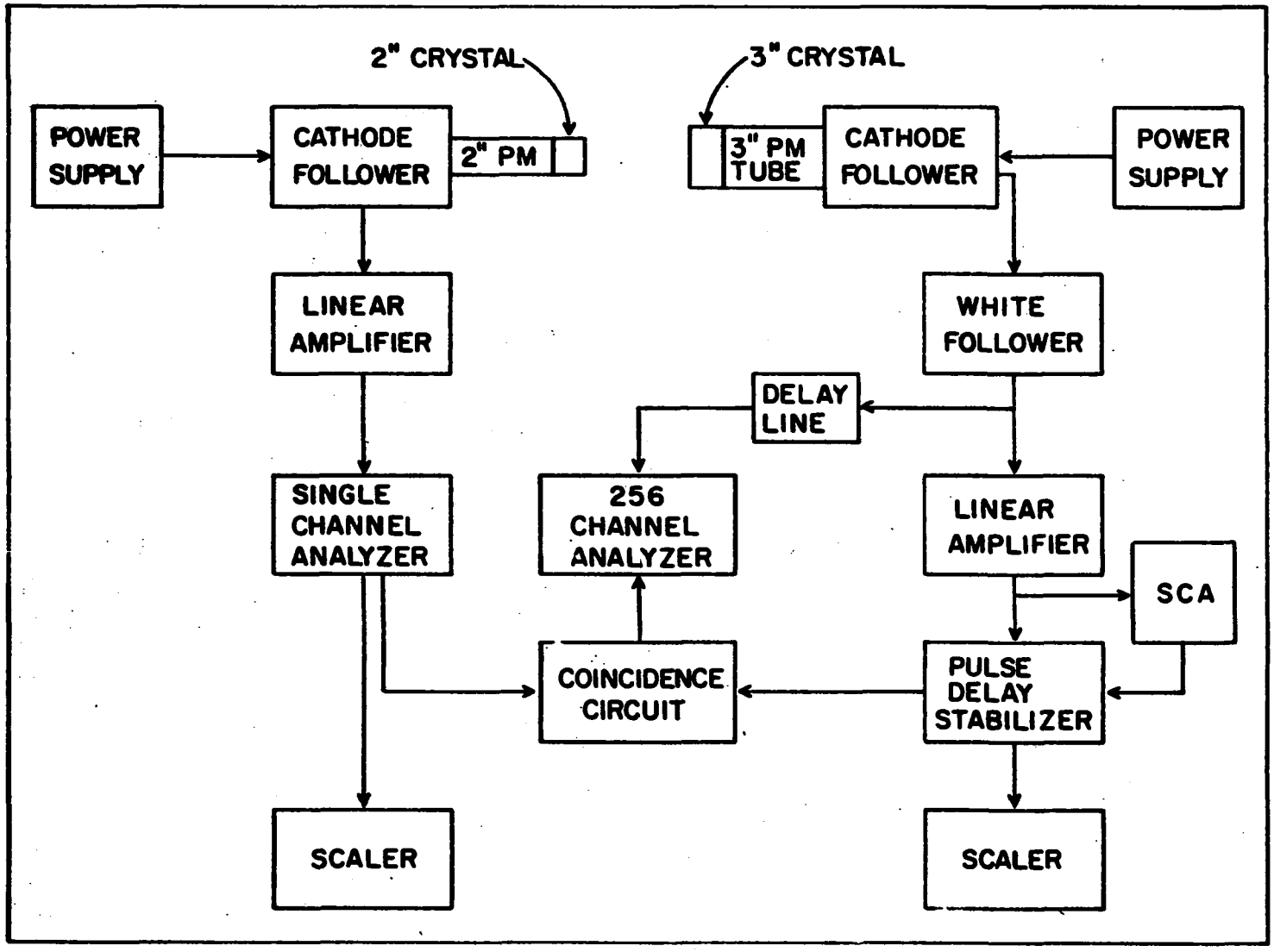
C. Scintillation Spectroscopy

A block diagram of the circuitry used to obtain the gamma-ray spectra taken in this investigation is shown in Figure 1. A cylindrical 2" x 2" Harshaw NaI(Tl) crystal optically coupled to a DuMont 6655 photomultiplier tube was associated with a cathode follower, linear amplifier (a Baird Associates Non-overloading Amplifier Model 215), power supply, single channel pulse height analyzer, and scaler to form a conventional single channel analyzer. This analyzer will be referred to as the gating counter.

A cylindrical 3" x 3" Harshaw NaI(Tl) crystal was coupled to a DuMont 6363 photomultiplier tube associated with a cathode follower, power supply, White follower, linear amplifier, single channel pulse height analyzer, pulse delay stabilizer, and scaler to form a modified single channel analyzer. This analyzer was operated in the integral mode during all coincidence runs.

The White follower transmits pulsed signals through long coaxial transmission lines without attenuation or distortion. This circuit was

Figure 1. Block diagram of scintillation spectroscopy equipment


2" CRYSTAL
3" CRYSTAL
POWER SUPPLY
CATHODE FOLLOWER
2" PM
3" PM TUBE
CATHODE FOLLOWER
POWER SUPPLY
LINEAR AMPLIFIER
WHITE FOLLOWER
DELAY LINE
SINGLE CHANNEL ANALYZER
256 CHANNEL ANALYZER
LINEAR AMPLIFIER
SCA
COINCIDENCE CIRCUIT
PULSE DELAY STABILIZER
SCALER
SCALER

a necessary part of the apparatus because of the relatively long lengths of coaxial cable required to interconnect the circuit elements fed by the three-inch detector.

The pulse delay stabilizer receives as an input all pulses being fed into a single channel pulse height analyzer and produces a delayed output pulse corresponding to each output pulse from the pulse height analyzer. There is normally a random fluctuation in the length of time between input and output pulses for a single channel pulse height analyzer, and for an analyzer operating in the integral mode, the size of the uncertainty in time may approach the magnitude of the resolving time used with coincidence circuits. The inclusion of the pulse delay stabilizer improves the precision of the measurements by reducing the uncertainty in the transit times of the signals being analyzed.

The multichannel analyzer used in this investigation was a Radiation Counter Laboratories Model 20611 256-channel analyzer.

The coincidence circuit was designed to provide an output pulse of the proper size and shape to open the gate in the multichannel analyzer when it received coincident pulses from the two single channel analyzers at its two input terminals. As the apparatus has been described so far, these permissive pulses from the coincidence circuit would arrive at the gate input several microseconds later than the pulse to be analyzed. A time delay was introduced with the delay line, which was an Advanced Electronics Type 2011 passive L-C network.

All of the items of equipment for which no manufacturer and model number were given were designed and constructed by the Electronics Shop

of the Institute for Atomic Research at Iowa State University.

D. Calibration of Equipment

The modified single channel analyzer was set to pass all pulses which entered it and did not need to be calibrated. If, for example, differential-differential coincidence operation was desired this analyzer would be calibrated.

The gating counter was calibrated in the following manner. The White follower was disconnected from the cathode follower of the three-inch crystal and connected to the output from the cathode follower of the two-inch crystal. With the equipment connected in this way, pulses from the two-inch crystal go through both the gating counter and modified single channel analyzer as well as directly into the multichannel analyzer through the delay line. Next, with the coincidence circuit turned off, the spectra of the sources of Tm^{173} (with which coincidence studies were to be made) were accumulated on the multichannel analyzer. The spectrum of a source of Yb^{175} which has strong gamma-rays at 53(Lu K-x ray), 114, 282, and 396 Kev was also accumulated to calibrate the two-inch crystal in terms of energy. Finally, the coincidence circuit was turned on. Now only those pulses which are allowed to pass by the gating counter are observed on the multichannel analyzer. So the position of the gate was observed directly on the multichannel analyzer.

It should be noted that when the equipment was connected as shown in Figure 1, a load equivalent to the modified single channel analyzer was connected to the output of the cathode follower of the two-inch crystal so

that the position of the gate would not shift from that found in the calibration.

The three-inch crystal was calibrated for energy assignments by running the spectrum of the Yb^{175} source with the equipment connected as in Figure 1 and the coincidence circuit off.

The operation of the coincidence circuit was checked by running several coincidence experiments on Yb^{175}. The results of these experiments agreed with the decay scheme of Yb^{175} as shown in Strominger, et al. (30).

XIV. RESULTS

The ratio of the yield of the activity from the irradiation of enriched ytterbium oxide (98.4% Yb^{174}) to that from the natural oxide (31.8% Yb^{174}) was $\sim$ 3 indicating that the activity is produced by a photonuclear reaction on Yb^{174}.

The activity is the major component produced in the irradiation of Yb^{174}. The activity decays by negatron emission indicating an excess of neutrons. The nuclides Yb^{173}, Yb^{172}, Yb^{171}, Tm^{171}, Tm^{170}, Er^{172}, and Er^{171} are either stable or have known decay characteristics different from the activity produced from Yb^{174}. Finally, the synchrotron was operated at 45 Mev during all the irradiations which rules out the production of significant quantities of nuclides produced by many-particle reactions. The above facts indicate that the activity is Tm^{173}.

The data on the half-life of Tm^{173} are not highly accurate due to the low counting rates and the interference of Tm^{172} and Yb^{175}. The average of several determinations is 8.0 hours with an estimated error of $\pm$ 0.5 hours.

Aluminum absorption measurements gave a value of 1.0 $\pm$ 0.1 Mev for the negatron end-point energy of the Tm^{173} decay.

The results of gamma-ray coincidence experiments on sources of Tm^{173} are shown in Figures 2-6. In each case the integral spectrum of Tm^{173} is shown for comparison. Figure 7 summarizes the results in a proposed decay scheme along with the previously known results from the decay of Lu^{173} (25, 26, 27) and coulomb excitation (28, 29). In Figure 2 is shown the spectrum of gamma-rays in coincidence with events above 360 kev. As can

Figure 2. Spectrum of coincidences with events above 360 kev.

_____ is integral spectrum

- - - - - is coincidence spectrum

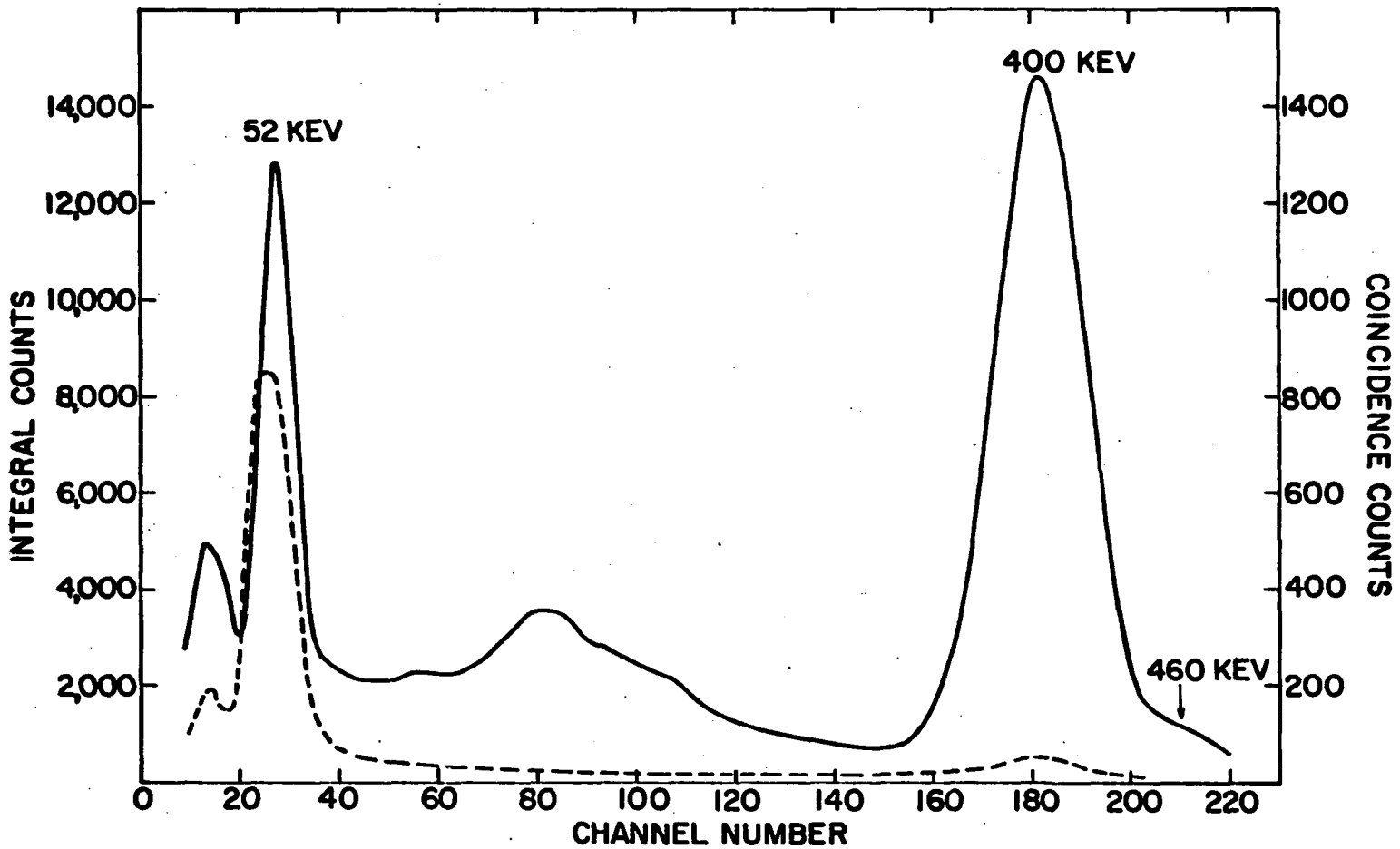

52 KEV
400 KEV
460 KEV
INTEGRAL COUNTS
COINCIDENCE COUNTS
CHANNEL NUMBER
14,000
12,000
10,000
8,000
6,000
4,000
2,000
1400
1200
1000
800
600
400
200
0
20
40
60
80
100
120
140
160
180
200
220

Figure 3. Spectrum of coincidences with events above 35 kev.

_____ is integral spectrum

- - - - - is coincidence spectrum

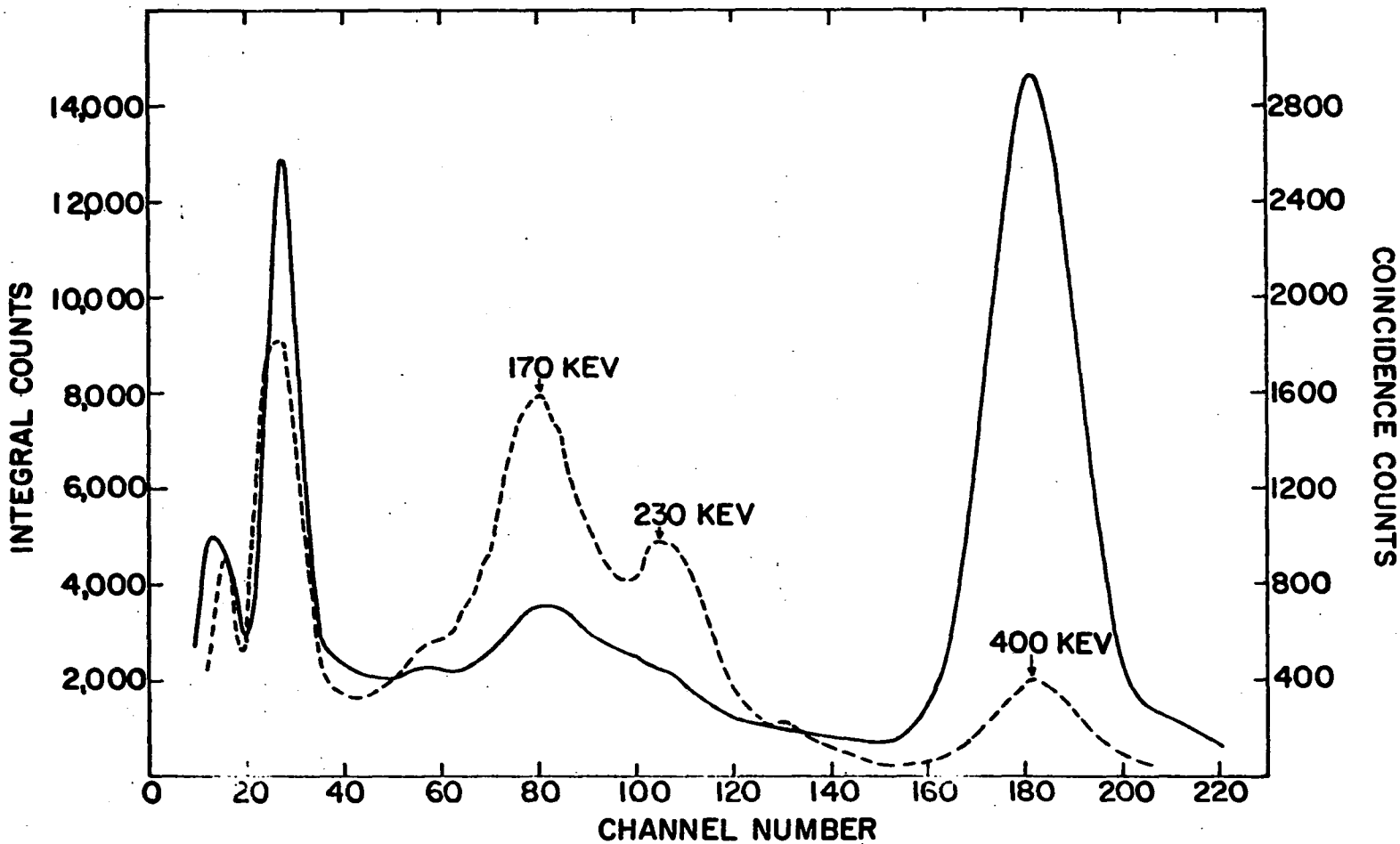

14,000
12,000
10,000
8,000
6,000
4,000
2,000
INTEGRAL COUNTS
2800
2400
2000
1600
1200
800
400
COINCIDENCE COUNTS
170 KEV
230 KEV
400 KEV
0
20
40
60
80
100
120
140
160
180
200
220
CHANNEL NUMBER

Figure 4. Spectrum of coincidences with events above 75 kev.

_____ is integral spectrum

- - - - - is coincidence spectrum

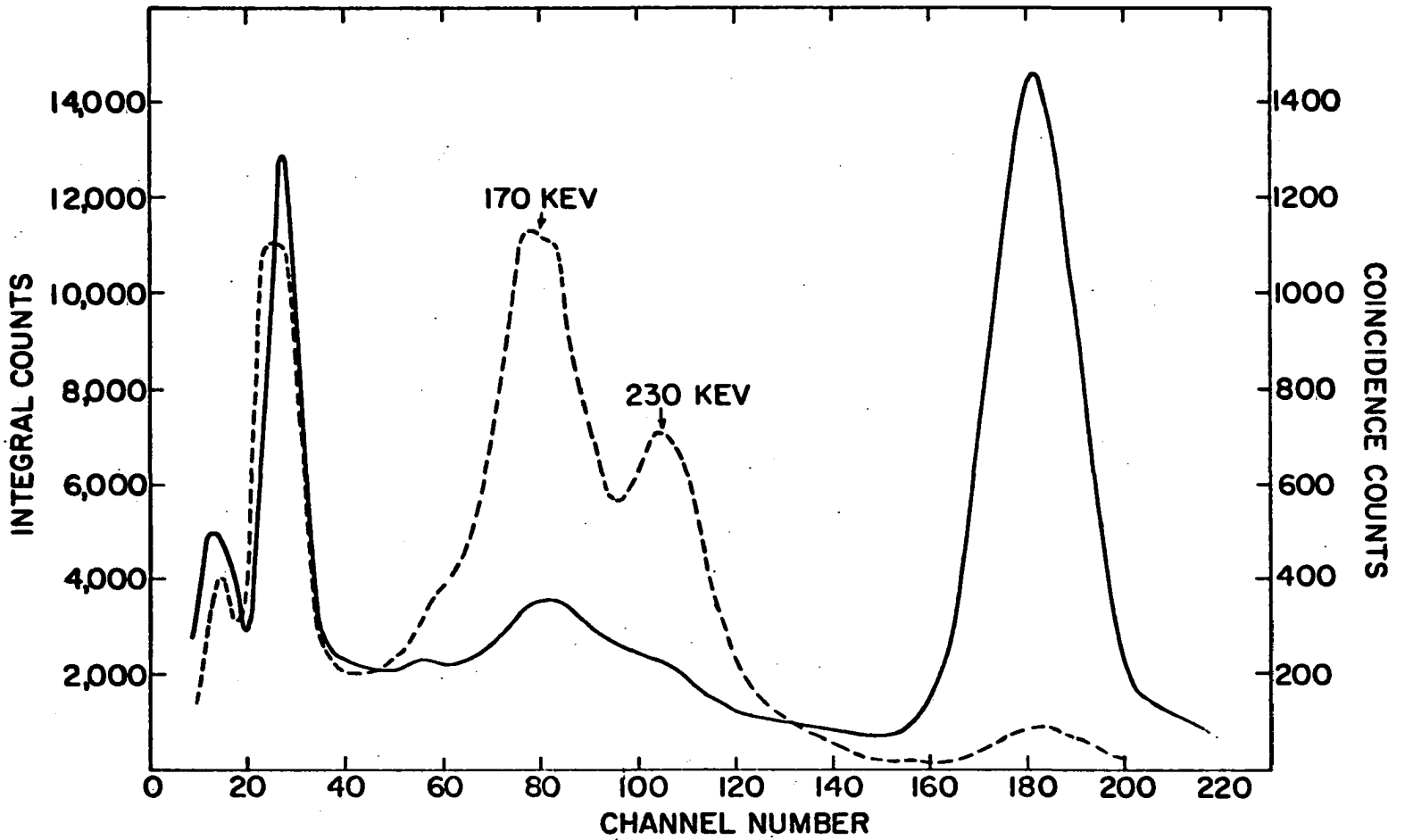

14,000
12,000
10,000
8,000
6,000
4,000
2,000
INTEGRAL COUNTS
1400
1200
1000
800
600
400
200
COINCIDENCE COUNTS
170 KEV
230 KEV
0
20
40
60
80
100
120
140
160
180
200
220
CHANNEL NUMBER

Figure 5. Spectrum of coincidences with the 230 kev. events

_____ is integral spectrum

- - - - - is coincidence spectrum

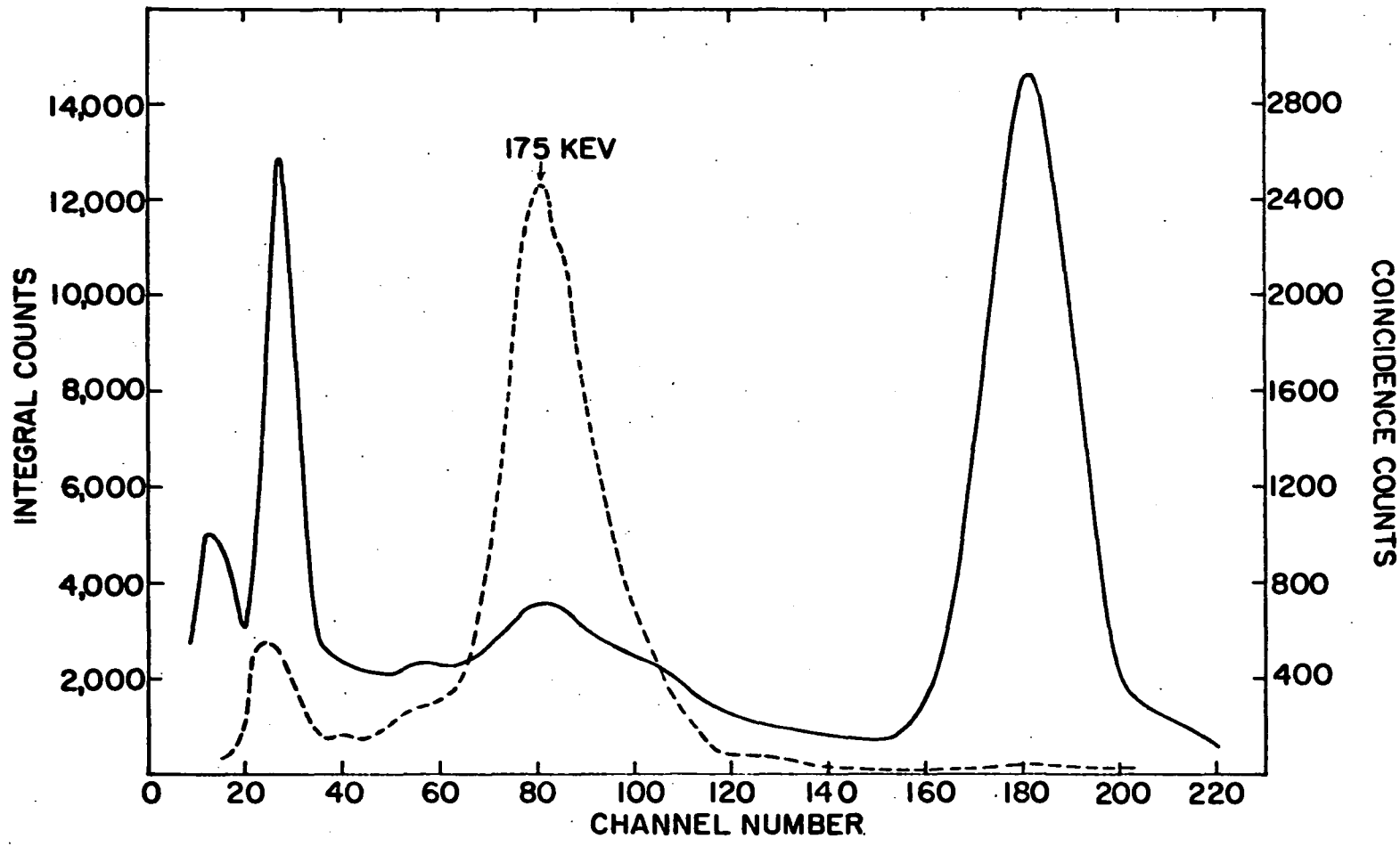

175 KEV
INTEGRAL COUNTS
COINCIDENCE COUNTS
CHANNEL NUMBER
14,000
12,000
10,000
8,000
6,000
4,000
2,000
2800
2400
2000
1600
1200
800
400
0
20
40
60
80
100
120
140
160
180
200
220

Figure 6. Spectrum of coincidences with the 170 kev. events

_____ is integral spectrum

- - - - - is coincidence spectrum

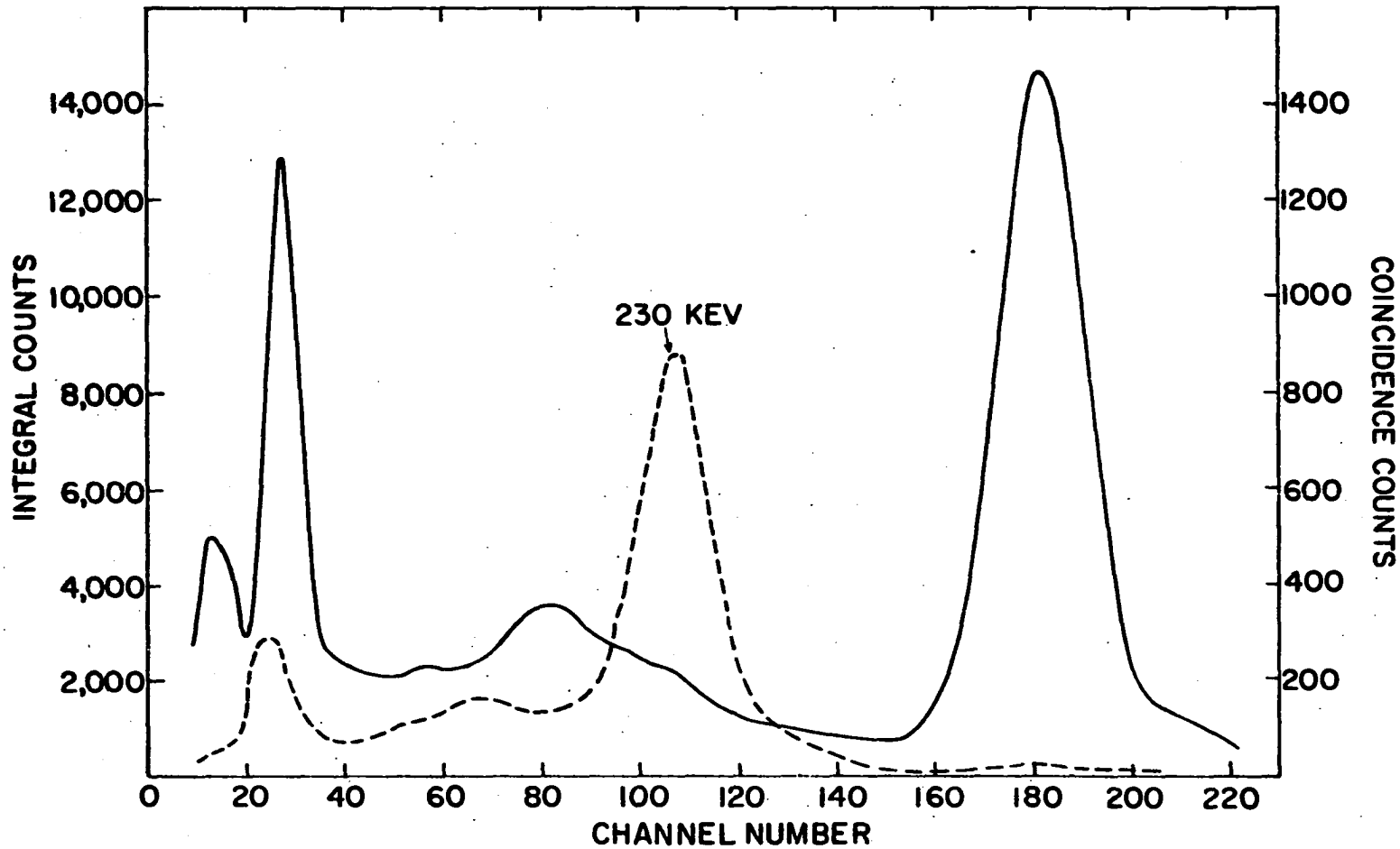
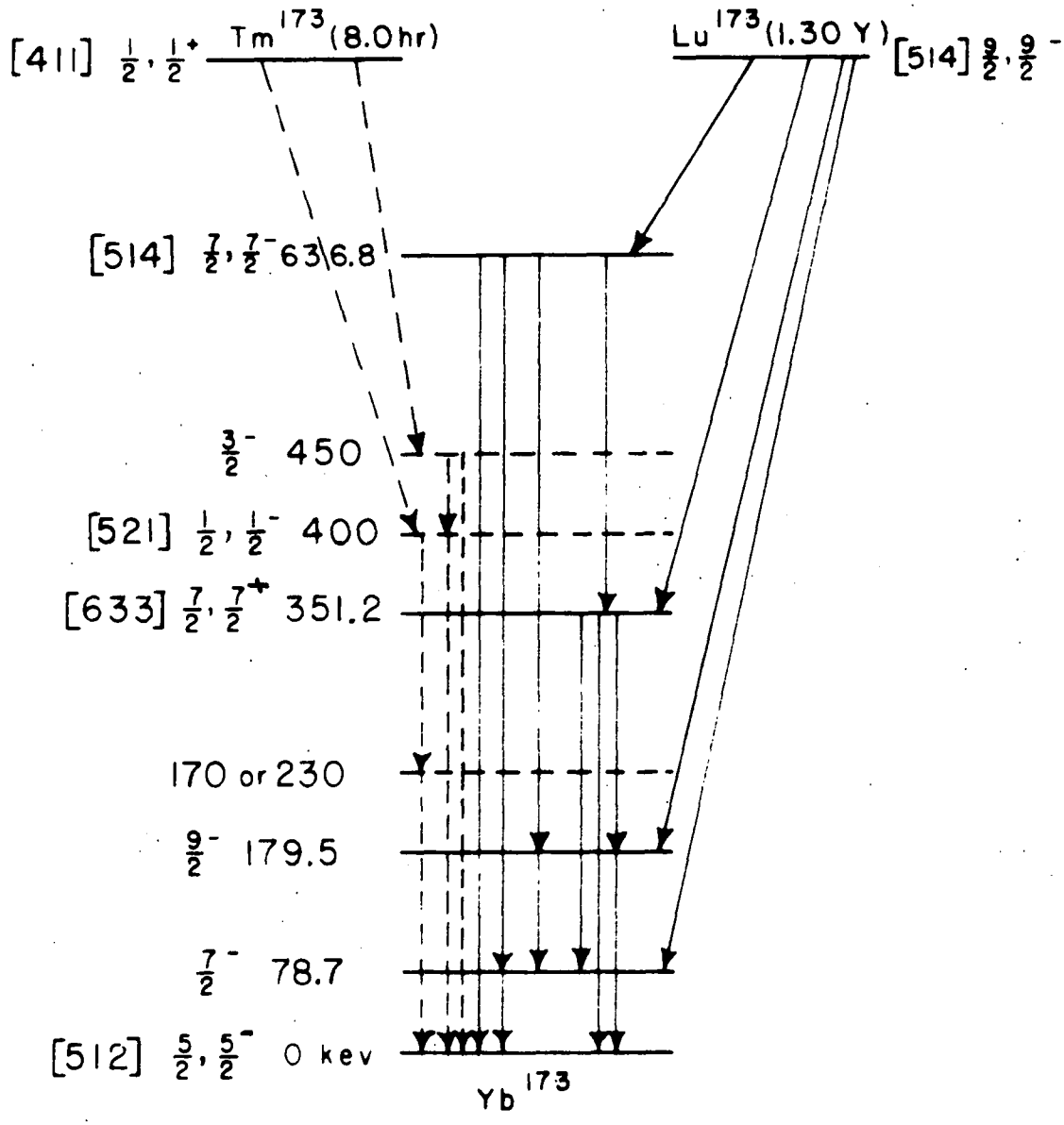

14,000
12,000
10,000
8,000
6,000
4,000
2,000
INTEGRAL COUNTS
1400
1200
1000
800
600
400
200
COINCIDENCE COUNTS
230 KEV
0
20
40
60
80
100
120
140
160
180
200
220
CHANNEL NUMBER

Figure 7. Energy levels of Yb^{173}

Solid lines indicate previously known levels

Dashed lines indicate new levels


[411] 1/2, 1/2+ Tm173 (8.0hr)
Lu173 (1.30 Y) [514] 9/2, 9/2-
[514] 7/2, 7/2- 63 \6.8
3/2- 450
[521] 1/2, 1/2- 400
[633] 7/2, 7/2+ 351.2
170 or 230
9/2- 179.5
7/2- 78.7
[512] 5/2, 5/2- 0 kev
Yb173

be seen from the integral spectrum there is a weak peak at 460 ± 10 kev. There are two possible interpretations of the 50 kev peak in the coincidence spectrum. The 50 kev peak could be Yb K-x-rays from a highly converted gamma-ray in coincidence with the 400 kev peak. The 50 kev peak could also be a 50 kev gamma-ray in coincidence with the 400 peak. The second interpretation would indicate a level at 450 kev assuming that the 400 kev gamma-ray is a transition from a level at 400 kev to the ground state of Yb^{173}. Since Tm^{173} has a ground state spin of 1/2, it is expected to decay to low-spin levels in Yb^{173}. From the level diagrams and data analysis of Mottelson and Nilsson (31) the lowest-lying low-spin level is Yb^{173} is expected to be the $\left[521\right]$ 1/2, 1/2-level. The analysis of Kuroyanagi, et al. (24) predicts that the first excited state of this rotational band, the 3/2-state, should be higher in energy by 45 kev. So with the second interpretation the 400 and 460 ± 10 kev gamma-rays can be consistently interpreted as transition from the $\left[521\right]$ 1/2, 1/2- and 3/2- levels at 400 and 460 ± 10 kev to the Yb^{173} ground state and the 50 kev coincidence gamma-ray as the transition between the 3/2- and 1/2- members of the rotational band.

In Figure 3 is shown the spectrum obtained by gating on all photons above 35 kev. The broad peak in the integral spectrum between 140 and 250 kev has split into two peaks at 170 and 230 kev. The 400 kev peak also appears in the coincidence spectrum. In Figure 4 which shows the coincidences with photons above 75 kev, the 400 kev peak has disappeared (except for chance coincidences). This agrees with the results shown in Figure 2, that the 400 kev peak is in coincidence with a 50 kev peak.

Figures 5 and 6 show that the 170 and 230 kev peaks are in coincidence. This indicates that there is a level at 170 or 230 kev but the position of the level cannot be decided from the data. The interpretation of this level in terms of asymptotic quantum numbers is difficult.

In Figure 7 these results are summarized and interpreted as new levels in Yb^{173} at 450, 400, and 170 or 230 kev. The assignment of the asymptotic quantum numbers to the ground state of Tm^{173} was made by analogy to the other odd-A thulium isotopes. Figure 7 shows beta branches to both the 400 and 450 kev levels but two beta rays with such close energies could not be resolved by aluminum absorption measurements. It is also possible that the 170 or 230 kev level is populated directly from the Tm^{173} ground state. If the level is at 170 kev this would explain the greater intensity of the 170 kev gamma-rays with respect to the 230 kev gamma-rays. One should note that none of the levels populated by Lu^{173} decay are populated by Tm^{173} decay because of the large spin change that would be involved.

XV. APPENDIX: ROTATIONAL MODEL OF THE RARE EARTH NUCLEI

The low lying energy levels of the rare earth nuclei are successfully explained with the rotational model proposed by Bohr and Mottelson (32, 33). The physical picture of the rotational model is an elliptically deformed nuclear shape which provides a non-spherical field in which the individual nucleons move. If the frequency of rotation of the deformed nucleus is slow compared with the frequency of individual particle motions (called intrinsic motions), then the rotation will not disturb the nucleon orbits in the distorted potential well. So the nucleus will have intrinsic levels generating rotational bands just like in molecules.

In the rotational model, the rotational eigenfunctions are the same as those for a symmetric top D^{I}_{MK} where $\vec{I}$ is the total angular momentum and M and K are the projections of $\vec{I}$ on the space z-axis and the axis of symmetry of the nucleus, respectively. Also one denotes the intrinsic angular momentum by $\vec{J}$ and its component along the body axis by Ω. To make detailed calculations it is also necessary to have the intrinsic wave functions. Nilsson (34) has carried out the most detailed calculations, using as a model a single particle moving outside on elliptically deformed core. For large deformations the intrinsic states may be approximately characterized by the so-called asymptotic quantum numbers. N, N_z, Λ where N is the total number of oscillation quanta, N_z the number of oscillator quanta along the symmetry axis, and Λ the component of single-particle, orbital angular momentum along the symmetry axis. In the literature, the quantum numbers of the asymptotic states are usually written in the following order $[N, N_z, \Lambda]\, K I^{\pi}$.

For example, a state with $N=5$, $N_z=1$, $\Lambda=2$, $K=5/2$, and negative parity would be written $[512]\,5/2^-$ where $I=K$, $K+1$, etc. Mottelson and Nilsson (31) have published level diagrams in which they have designated the levels by the asymptotic quantum numbers.

XVI. LITERATURE CITED

1. Dirac, P. A. M., Proc. Roy. Soc. (London) A117, 610 (1928).
2. Dirac, P. A. M., Proc. Roy. Soc. (London) A155, 447 (1936).
3. Fierz, M., Helv. Phys. Acta 12, 3 (1939).
4. Fierz, M. and Pauli, W., Proc. Roy. Soc. (London) A 173, 211 (1939).
5. Kemmer, N., Proc. Roy. Soc. (London) A173, 91 (1939).
6. Duffin, R. J., Phys. Rev. 54, 1114 (1938).
7. Bhabha, H. J., Rev. Mod. Phys. 17, 200 (1945).
8. Jacob, M. and Wick, G. C., Am. Phys. (N. Y.) 7, 404 (1959).
9. Foldy, L. L. and Wouthuysen, S. A., Phys. Rev. 78, 29 (1950).
10. Taketani, M. and Sakata, S., Proc. Phys. Math. Soc. (Japan) 22, 757 (1939).
11. Case, K. M., Phys. Rev. 95, 1323 (1954).
12. Foldy, L. L., Phys. Rev. 102, 568 (1956).
13. Guth, E., Bull. Am. Phys. Soc. 7, 576 (1962).
14. Bollini, C. G. and Giambiagi, J. J., Nuovo Cimento 21, 107 (1961).
15. Good, R. H., Jr. and Rose, M. E., Nuovo Cimento 24, 864 (1962).
16. Hammer, C. L. and Good, R. H., Jr., Phys. Rev. 108, 882 (1957).
17. Møller, C., The Theory of Relativity, (Oxford at the Clarendon Press, London, England, 1952).
18. Hausdorff, F., Leipzig, Ber. Ges. Wiss. Math. Phys. Kl. 58, 19 (1906).
19. Edmonds, A. R., Angular Momentum in Quantum Mechanics, (Princeton Univ., New Jersey, 1957).
20. Fradkin, D. M. and Good, R. H., Jr., Rev. Mod. Phys. 33, 343 (1961).
21. Good, R. H., Jr., Phys. Rev. 105, 1914 (1957).
22. Good, R. H., Jr., Nuovo Cimento 24, 713 (1962).

23. Hammer, C. L. and Good, R. H., Jr., Phys. Rev. 111, 342 (1958).

24. Kuroyanagi, T., Yuta, H., Takahashi, K., and Morinaga, H., J. Phys. Soc. (Japan) 16, 2393 (1961).

25. Dillman, L. T., Henry, R. W., Gove, N. B., and Becker, R. A., Phys. Rev. 113, 635 (1959).

26. Bichard, J. W., Mihelich, J. W., and Harmatz, B., Phys. Rev. 116, 720 (1959).

27. Wilson, R. G. and Pod, M. L., Phys. Rev. 117, 807 (1960).

28. Temmer, G. M. and Heydenburg, N. P., Phys. Rev. 100, 150 (1955).

29. Elbek, B., Nielsen, K. O., and Olesen, M. C., Phys. Rev. 108, 406 (1957).

30. Strominger, D., Hollander, J. M., and Seaborg, G. T., Rev. Mod. Phys. 30, 585 (1959).

31. Mottelson, B. R. and Nilsson, S. G., Dan. Mat. fys. Skr. 1, No. 8 (1959).

32. Bohr, A., Dan. Mat. fys. Medd. 26, No. 14 (1952).

33. Bohr, A. and Mottelson, B. R., Dan. Mat. fys. Medd. 27, No. 16 (1953).

34. Nilsson, S. G., Dan. Mat. fys. Medd. 29, No. 16 (1955).

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